Mappings into $\mathbf{C}^N$

Howard Jacobowitz

Rutgers University - Camden

References

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$$\mathbf{C}^N = (\mathbf{R}^{2N}, J)$$

$M^n \subset (\mathbf{R}^{2N}, J)$ is a totally real submanifold if

$$TM \cap JTM = \{0\}.$$

A map $f : M^n \to \mathbf{R}^{2N}$ is a totally real immersion if

$$\mathrm{rank}(f_x, Jf_x) = 2n$$

at each point of M.

Theorem

a. Any map $f : M^n \to \mathbf{C}^N$ may be approximated by a totally real embedding, provided $N \geq [\frac{3n}{2}]$.
b. For each n there exists some M^n that does not have any totally real immersion into $\mathbf{C}^N$ for $N = [\frac{3n}{2}] - 1$.

The 1-jet bundle of maps of $M \to \mathbf{C}^N$

$$J(M, \mathbf{C}^N) = \{(p, j^1(f)(p)) \mid p \in M,\ f : \mathcal{O}_p \to \mathbf{R}^{2N}\}.$$

In local coordinates

$$j^1(f) = (p, f(p), \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$$

and

$$J(M, \mathbf{C}^N) = \{(p, q, a^1, \dots, a^N)\}.$$

The set to avoid

$$\Sigma = \{(p, q, a) \mid \mathrm{rank}(a, Ja) < 2n\} \subset J(M, \mathbf{C}^N).$$

Σ is a stratified manifold.

$$\Sigma = \bigcup_{k=1}^{n} \Sigma_k.$$

where

$$\Sigma_k = \{(p, q, a) \mid \mathrm{rank}(a, Ja) = 2n - 2k\}.$$

Σ_k is a locally closed manifold,

$$\overline{\Sigma_k} = \bigcup_{i=k}^{n} \Sigma_i.$$

Proof of part a. of the Theorem

a. Any map $f : M^n \to \mathbf{C}^N$ may be approximated by a totally real embedding, provided $N \geq [\frac{3n}{2}]$.
The codimension of Σ_1 is $2(N - n + 1)$.
If $n < 2(N - n + 1)$ then for any $f : M \to \mathbf{C}^N$ the image $j^1(f)(M) \subset J^1(M, \mathbf{C}^N)$ may be perturbed to not intersect Σ.
Thom Transversality theorem implies that there is some $g : M \to \mathbf{C}^N$ that approximates f and satisfies

$$j^1(g) \cap \Sigma = \emptyset.$$

This proves the first part of the Theorem.

Independent maps

$f : M^n \to \mathbf{C}^N$ is an independent map if

$$df_1 \wedge \cdots \wedge df_N(p) \neq 0$$

for all $p \in M$.

Theorem

a. Any map $f : M^n \to \mathbf{C}^N$ may be approximated by an independent map, provided $N \leq [\frac{n+1}{2}]$.
b. For each n there exists some M^n that does not have any independent map into $\mathbf{C}^N$ for $N > [\frac{n+1}{2}]$.

Proof of part a. of the Theorem

Any map $f : M^n \to \mathbf{C}^N$ may be approximated by an independent map, provided $N \leq [\frac{n+1}{2}]$.
We now write $j^1(f) = (p, q, f_{x_1}, \ldots, f_{x_n})$ and, in local coordinates,

$$\Sigma = \{(p, q, \alpha_1, \ldots, \alpha_n) \mid \mathrm{rank}\ \alpha < N\}.$$

Σ is a stratified manifold of codimension $2(n - N + 1)$.
If $n < 2(n - N + 1)$ then any map

$$F : M \to \mathbf{C}^N$$

may be approximated by an independent map.

Optimality Results

Lemma

If M has a totally real immersion into $\mathbf{C}^N$ then there exists a bundle Q of rank N-n such that

$$(\mathbf{C} \otimes TM) \oplus Q$$

is trivial.

Lemma

If M has an independent map into $\mathbf{C}^N$ then there exists a bundle of rank n-N such that

$$(\mathbf{C} \otimes TM) = N\varepsilon \oplus Q$$

where $N\varepsilon$ is the trivial bundle of rank N.

Proof of the first lemma

Let $M \subset \mathbf{C}^N$ be a totally real submanifold. Define

$$\Phi : (\mathbf{C} \otimes TM) \to T^{1,0}\mathbf{C}^N$$

by

$$\Phi(v) = v - iJv.$$

Φ is injective because

$$\Phi(\xi + i\eta) = 0 \Rightarrow J\eta \in TM.$$

So there exists some Q such that

$$(\mathbf{C} \otimes TM) \oplus Q$$

is trivial.
The proof of the second lemma is similar.

The optimality results are thus reduced to properties of bundles.

Lemma

There exist manifolds of dimensions $2k$ and $2k+1$ such that if

$$(\mathbf{C} \otimes TM) \oplus Q$$

is trivial, then the rank of Q is at least k.

Lemma

There exists a manifold of dimension $2k$ such that if

$$(\mathbf{C} \otimes TM) = r\varepsilon \oplus Q$$

then the $r < k$. There exists a manifold of dimension $2k+1$ such that if

$$(\mathbf{C} \otimes TM) = r\varepsilon \oplus Q$$

then $r < k+1$.

The examples use the dimension modulo 4

Let

$$M^{4k} = \mathbf{CP}^2 \times \cdots \times \mathbf{CP}^2 = (\mathbf{CP}^2)^{\times k}$$

and

$$\begin{aligned} M^{4k+1} &= M^{4k} \times S^1 \\ M^{4k+2} &= M^{4k} \times \mathbf{RP}^2 \\ M^{4k+3} &= M^{4k} \times \mathbf{RP}^2 \times S^1. \end{aligned}$$

Theorem

If M^n has a totally real immersion into $\mathbf{C}^N$, then $N \geq [\frac{3n}{2}]$. If M^n has an independent map into $\mathbf{C}^N$ then $N \leq [\frac{n+1}{2}]$.

Proofs for M^{4k+3}

Totally real embeddings:

We need to show that if

$$(\mathbf{C} \otimes TM) \oplus Q$$

is trivial, then $\mathrm{rank} Q \geq 2k+1$. We have

$$c((\mathbf{C} \otimes TM) \oplus Q) = c(\mathbf{C} \otimes TM) \smile c(Q) = 1.$$

Let a be the first Chern class of the hyperplane bundle of $\mathbf{CP^2}$.

$$\begin{aligned} c(\mathbf{C} \otimes T\mathbf{CP^2}) &= c(T^{1,0}) \smile c(T^{0,1}) \\ &= (1+a)^3(1-a)^3 \\ &= 1-3a^2. \end{aligned}$$

Let

$$f_j : M^{4k+3} = (\mathbf{CP}^2)^{\times k} \times \mathbf{RP}^2 \times S^1 \to \mathbf{CP}^2$$

and

$$a_j = f_j^* a.$$

Let b_1 be the pull-back of the first Chern class of the tautological bundle on $\mathbf{RP}^2$.

The proof that rank $Q \geq 2k+1$

$$c(\mathbf{C} \otimes TM^{4k+3}) = (1-3a_1^2)\cdots(1-3a_k^2)(1+b_1).$$

$$\begin{aligned} a^3 = 0 \quad &\Rightarrow \quad (1-3a_j^2)^{-1} = 1+3a_j^2 \\ b^2 = 0 \quad &\Rightarrow \quad (1-b_1)^{-1} = 1+b_1. \end{aligned}$$

So

$$c(Q) = (1+3a_1^2)\cdots(1+3a_k^2)(1+b_1).$$

Since

$$3^k a_1^2 \cdots a_k^2 b_1 \neq 0,$$

we have that

$$c_{2k+1}(Q) \neq 0$$

which implies that the rank of Q is at least $2k+1$.

Independent mappings:
We need to show that if

$$(\mathbf{C} \otimes TM) = N\varepsilon \oplus Q$$

then $N \leq [\frac{n+1}{2}]$.
This is equivalent to

$$\mathrm{rank} Q \geq 2k + 1.$$

Since

$$c(\mathbf{C} \otimes TM) = c(Q)$$

it follows that

$$c_{2k+1}(Q) = -(-3)^k a_1^2 \cdots a_k^2 b_1 \neq 0,$$

and we again have

$$c_{2k+1}(Q) \neq 0.$$

A slight improvement

Note that M^{4k+2} and M^{4k+3} are non-orientable.

Theorem

Any orientable manifold M^n, for $n = 4k + 2$, has a totally real embedding into $\mathbf{C}^N$, for $N = [\frac{3n}{2}] - 1$.

Is a similar improvement possible for $n = 4k + 3$?

The proof of the theorem consists of three parts.

- Euler characteristic
- " K-theory "
- Gromov's h- principle

The Euler characteristic

Lemma

Let M be an orientable manifold of dimension $4k+2$ and Q_1 be a complex line bundle of rank $r = 2k+1$. If $(\mathbf{C} \otimes TM) \oplus Q_1$ is trivial, then there exists some Q_2 such that

$$Q_1 = Q_2 \oplus \varepsilon.$$

Proof

It suffices to show

$$2c_r(Q_1) = 0.$$

Note that r is an odd integer.

From $c((\mathbf{C} \otimes TM) \oplus Q_1) = 1$, we have

$$c_r(Q_1) = \Sigma a_J c_{j_1}(\mathbf{C} \otimes TM) \cdots c_{j_m}(\mathbf{C} \otimes TM)$$

with at least one j_s being odd.
For any manifold

$$(\mathbf{C} \otimes TM) \cong \overline{(\mathbf{C} \otimes TM)}$$

and this implies

$$2c_j(\mathbf{C} \otimes TM) = 0$$

for each odd j.

$$2c_r(Q_1) = 0.$$

Application to our case

M^n, for $n = 4k + 2$, has a totally real embedding into $\mathbf{C}^N$, $N = 6k + 3$. So there exists some Q_1 with

$$(\mathbf{C} \otimes TM) \oplus Q_1 = N\varepsilon$$

and

$$\mathrm{rank} Q_1 = 2k + 1.$$

Thus there exists some Q_2 of rank $2k$ with

$$(\mathbf{C} \otimes TM) \oplus Q_2 \oplus \varepsilon = N\varepsilon.$$

We would like to " subtract " ε from both sides.

K Theory

$\mathrm{Vect}_k(M)$ is the set of isomorphism classs of rank k complex vector bundles over M.

$$\mathrm{Vect}_k(M) \to \mathrm{Vect}_{k+1}(M)$$

is the map $B \to B \oplus \varepsilon$.

Lemma

(Husemöller) This map is bijective if $k \geq [\frac{\dim M+1}{2}]$.

$$(\mathbf{C} \otimes TM) \oplus Q_2$$

and

$$(N-1)\varepsilon$$

both map to

$$N\varepsilon = (6k+3)\varepsilon.$$

So

$$(\mathbf{C} \otimes TM) \oplus Q_2 = (N-1)\varepsilon.$$

h-principle

From

$$(\mathbf{C} \otimes TM) \oplus Q_2 = (N-1)\varepsilon = (6k+2)\varepsilon$$

we may conclude that M^{4k+2} has a totally real embedding into $\mathbf{C}^{6k+2}$.

Remark

This result is optimal. The orientable manifold

$$M^{4k+2} = (\mathbf{CP}^2)^{\times k} \times S^2$$

does not have a totally real immersion into $\mathbf{C}^N$, for $N = 6k + 1$.

A similar improvement for independent mappings

Theorem

Any orientable manifold M^n, $n = 4k+2$, has an independent map into $\mathbf{C}^N$, $N = [\frac{n+1}{2}] + 1 = 2k+2$.

We need

$$(\mathbf{C} \otimes TM) = (2k+2)\varepsilon \oplus Q_1$$

and we have

$$(\mathbf{C} \otimes TM) = (2k+1)\varepsilon \oplus Q_2.$$

We will use the Euler characteristic and the h-principle.

$c_{2k+1}(\mathbf{C} \otimes TM) = 0$ (since $2k+1$ is an odd number)

$\Rightarrow c_{2k+1}(Q_2) = 0$

$\Rightarrow Q_2 = Q \oplus \varepsilon$

$\Rightarrow (\mathbf{C} \otimes TM) = (2k+2)\varepsilon \oplus Q$

$\Rightarrow M$ has an independent map into $\mathbf{C}^{2k+2}$.

Again this is optimal.

Stein Manifolds

Theorem (Bishop, Eliashberg-Gromov, Shürmann, Forstnerič)

For $n > 1$, every n-dimensional Stein manifold has a proper holomorphic immersion into $\mathbf{C}^N$, $N = [\frac{3n+1}{2}]$ and a proper holomorphic embedding into $\mathbf{C}^N$, $N = 1 + [\frac{3n}{2}]$.

These dimensions are optimal, at least when n is even (Forster, 1970). New set of examples for optimality:

$$M^{4k} = \mathbf{CP}^2 \times \cdots \times \mathbf{CP}^2 = (\mathbf{CP}^2)^{\times k}$$

and

$$\begin{aligned} M^{4k+1} &= M^{4k} \times S^1 \\ M^{4k+2} &= M^{4k} \times \mathbf{RP}^2 \\ M^{4k+3} &= M^{4k} \times \mathbf{RP}^2 \times S^1. \end{aligned}$$

M^n as above and X^n a Stein neighborhood in the complexification of M. If $f : X^n \to \mathbf{C}^N$ is a holomorphic immersion, then f restricted to M is a totally real immersion.
X^n does not immerse into $[\frac{3n}{2}] - 1$.

Holomorphic Submersions

Theorem (Forstnerič)

Every n-dimensional Stein manifold has a holomorphic submersion into $\mathbf{C}^N, N = [\frac{n+1}{2}]$.

This dimension is optimal (Forster, 1967).
New set of examples for optimality: If

$$f : X^n \to \mathbf{C}^N$$

is a submersion, then

$$f : M^n \to \mathbf{C}^N$$

is an independent map.
X^n does not have a holomorphic submersion into $\mathbf{C}^N$ for $N = [\frac{n+1}{2}] - 1$.

THANK YOU

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