Left-Invariant CR and Pseudo hermitian Structures on S^3

Howard Jacobowitz
Rutgers University Camden
January 4, 2016

CR Structures

A **CR structure** on M^3 is a two-plane distribution $H \subset TM$ and a complex structure on each fiber.

$$J : H \to H \text{ with } J^2 = -I.$$

We denote this structure by (M, H, J).
It is often useful to extend J by complex linearity to a map

$$J : \mathbf{C} \otimes H \to \mathbf{C} \otimes H.$$

Then J is completely determined by the eigenspace corresponding to the eigenvalue i (or to the eigenvalue -i). So a CR structure is just as well given by a complex line bundle $B \subset \mathbf{C} \otimes H$,

$$B \cap \overline{B} = \{0\}.$$

It will be useful to work with the dual formulation. Choose some θ such that $\theta^\perp = H$. Choose some θ^1 such that

$$X \in H \implies \theta^1(X + iJX) = 0 \text{ and } \theta^1(X - iJX) \neq 0.$$

(θ, θ^1) is called a CR coframe. The CR structure is **strictly pseudoconvex** if $\theta \wedge d\theta \neq 0$. So θ is a contact form and H is a contact distribution. A normalized coframe (θ, θ^1) satisfies

$$d\theta = i\theta^1 \wedge \overline{\theta^1}.$$

(H, J) uniquely defines the CR structure. are not unique.

$$\begin{aligned} \tilde{\theta} &= r\theta \\ \tilde{\theta^1} &= \alpha\theta^1 \end{aligned}$$

with constants r real and α complex, $|\alpha|^2 = r > 0$ is also a normalized coframe.

A **pseudo-hermitian structure** is a strictly pseudoconvex (H, J) and a choice of θ. If (θ, θ^1) is a normalized coframe, then the only other normalized choices are

$$\begin{aligned} \tilde{\theta} &= \theta \\ \tilde{\theta^1} &= \lambda\theta^1 \end{aligned}$$

with $|\lambda| = 1$.

Given two CR structures (M, H, J) and $(M, \tilde{H}, \tilde{J})$ a diffeomorphism $F : M \to M$ is a **CR diffeomorphism** if it preserves the contact distribution and the J-operator. That is

$$F_* \circ J = \tilde{J} \circ F_*.$$

In terms of choices of coframes we are requiring

$$F^* \tilde{\theta} = s\theta$$

$$F^* \tilde{\theta^1} = \gamma\theta^1 + \delta\theta$$

with s real, γ and δ complex $s \neq 0$, and $\gamma \neq 0$.

Given two pseudo-hermitian structures, say $\{\theta, \theta^1\}$ and $\{\theta, \tilde{\theta^1}\}$ and a diffeomorphism $F : M^3 \to M^3$, we say that the two pseudo-hermitian structures are equivalent, and that F is a **pseudo-hermitian diffeomorphism** if

$$F^*(\theta) = \theta$$

and

$$F^*(\tilde{\theta^1}) = \gamma\theta^1 + \delta\theta.$$

The Standard Structures

The **standard CR structure** on the three sphere S^3 is the one it inherits as a submanifold of $\mathbf{C}^2$.

$$H = TS^3 \cap JTS^3.$$

H is called the **standard contact distribution**.
Restricting $\theta_0 = -i(\overline{z}dz + \overline{w}dw)$ to S^3 gives the **standard pseudo-hermitian structure**.

The induced CR structure on S^3 can be given by

$$\begin{aligned} \theta_0 &= -i(\overline{z}dz + \overline{w}dw) \\ \theta_0^1 &= wdz - zdw \end{aligned}$$

where these forms are restricted to S^3.
Note

$$\begin{aligned} d\theta_0 &= i\theta_0^1 \wedge \overline{\theta_0^1} \\ d\theta_0^1 &= \theta_0^1 \wedge \omega \\ \omega &= -2i\theta_0 \\ d\omega &= 2\theta_0^1 \wedge \overline{\theta_0^1} \end{aligned}$$

“ The pseudo-hermitian curvature equals two.”

The Webster Connection

Theorem

Let (θ, θ^1) be a pseudo-hermitian coframe. There exist unique functions R, A, and V, and an unique one-form ω, so that

$$\begin{aligned} d\theta &= i\theta^1 \wedge \overline{\theta^1} \\ d\theta^1 &= \theta^1 \wedge \omega + A\theta \wedge \overline{\theta^1} \\ \omega &= -\overline{\omega} \\ d\omega &= R\theta^1 \wedge \overline{\theta^1} + 2i\Im(V\overline{\theta^1}) \wedge \theta. \end{aligned}$$

Further, if θ^1 is replaced by $\boldsymbol{\theta}^{\mathbf{1}} = \lambda\theta^1$, $|\lambda| = 1$, then

$$\mathbf{R} = R, \quad \mathbf{A} = \lambda^2 A, \quad \mathbf{V} = \lambda V, \quad \boldsymbol{\omega} = \omega - \lambda^{-1} d\lambda.$$

Recall from previous slide that $R = 2$, $A = 0$, $V = 0$ for the standard pseudo-hermitian structure.

Cartan structural equations

Let ϕ and ϕ_1 be one-forms with ϕ real giving the CR structure:

1. $\phi^\perp = H$,
2. $J\phi_1 = i\phi_1$,
3. $d\phi = i\phi_1\overline{\phi_1}$.

Theorem

There exist unique one-forms ϕ_2, ϕ_3, ϕ_4 and unique functions R_C and S such that

1. *ϕ_2 is imaginary and ϕ_4 is real,*
2. $d\phi_1 = -\phi_1\phi_2 - \phi\phi_3$,
3. $d\phi_2 = 2i\phi_1\overline{\phi_3} + i\overline{\phi_1}\phi_3 - \phi\phi_4$,
4. $d\phi_3 = -\phi_1\phi_4 - \overline{\phi_2}\phi_3 - R_C\phi\overline{\phi_1}$,
5. $d\phi_4 = i\phi_3\overline{\phi_3} + (S\phi_1 + \overline{S\phi_1})\phi$.

If we replace ϕ by $\psi = |\nu|^2\phi$ and ϕ_1 by $\psi_1 = \nu\phi_1$ with a constant ν then the forms

$$\psi_2 = \phi_2, \quad \psi_3 = \frac{1}{\overline{\nu}}\phi_3, \quad \psi_4 = \frac{1}{|\nu|^2}\phi_4$$

satisfy the equations in the Theorem with R and S replaced by

$$R = \frac{R}{|\nu|^2\overline{\nu}^2} \text{ and } S = \frac{S}{|\nu|^2\nu}.$$

R and S are relative invariants.

We want to choose a multiple of ϕ and a corresponding multiple of ϕ_1 so that $R(x) \equiv 1$.

Corollary

If $R(p) \neq 0$, there are precisely two choices of (ϕ, ϕ_1) such that in a neighborhood of p

1. *(ϕ, ϕ_1) give the CR structure,*
2. *$d\phi = i\phi_1\overline{\phi_1}$, and*
3. *$R \equiv 1$.*

The eight parameter space of choices is cut down to only two choices.

If we denote one choice by (ω, ω_1), then the other choice is $(\omega, -\omega_1)$.
We set $\phi = \omega$ and $\phi_1 = \omega_1$ and apply the theorem to obtain $\phi_2,\ \phi_3,$ and ϕ_4.

$$\phi_2' = \phi_2, \quad \phi_3' = -\phi_3, \text{ and } \phi_4' = \phi_4.$$

So the curvature R is a pseudo-hermitian invariant and the torsion A is a relative invariant. NEED CLARIFY THIS

The group structure

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\} \subset O(4).$$

Identify $SU(2)$ with S^3. $SU(2)$ acts on S^3 on the left.

Facts

- Every left-invariant 2-plane distribution is contact.
- The standard contact distribution is left-invariant.

Start by describing all left-invariant CR structures with the standard contact distribution.

$$B \subset \mathbf{C} \otimes H \subset \mathbf{C} \otimes TS^3.$$

$$B = \{Z \in \mathbf{C} \otimes TS^3 \mid \theta_0, (\alpha\theta_0^1 + \beta\overline{\theta_0^1})Z = 0\}.$$

Assume $\alpha \neq 0, \quad |\beta/\alpha| \neq 1.$

$$B = \{\theta_0, \theta_0^1 + \mu\overline{\theta_0^1}\}^\perp$$

satisfies $B \cap \overline{B} = \{0\}$.

Let $\theta(\mu) = \theta_0^1 + \mu\overline{\theta_0^1}$.

Theorem

1. *The left-invariant CR structures* $(\theta_0, \theta^1(\mu))$ *and* $(\theta_0, \theta^1(\mu'))$ *are equivalent if and only if either* $|\mu| = |\mu'|$ *or* $|\mu| = |\mu'|^{-1}$.
2. *The left-invariant pseudo-hermitian structures* $(a\theta_0, \theta^1(\mu))$ *and* $(a'\theta_0, \theta^1(\mu'))$ *are equivalent if and only if* $a = a'$ *and* $|\mu| = |\mu'|$.
3. *Any left-invariant CR structure of pseudo-hermitian structure is equivalent to one with the standard contact distribution.*

The third statement may be proved by group theory, but analysis yields more information.

We need the Webster and Cartan curvatures for $(\theta_0, \theta^1(\mu)$.

$$d\theta = i\theta^1 \wedge \overline{\theta^1}$$

$$d\theta^1 = \theta^1 \wedge \left(-2i\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right)\right)\theta - \frac{4i\mu}{1-|\mu|^2}\theta \wedge \overline{\theta^1}.$$

We need the Webster and Cartan curvatures for $(\theta_0, \theta^1(\mu)$.

$$d\theta = i\theta^1 \wedge \overline{\theta^1}$$

$$d\theta^1 = \theta^1 \wedge \left(-2i\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right)\right)\theta - \frac{4i\mu}{1-|\mu|^2}\theta \wedge \overline{\theta^1}.$$

So

$$A = -\frac{4i\mu}{1-|\mu|^2}$$

and

$$R = 2\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right).$$

Note that $RA \neq 0$ for $\mu \neq 0$.