HOLOMOROPHIC SECTIONS OF POWERS OF A LINE BUNDLE

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1. Introduction

Under various hypotheses and contexts we have

$$\dim \Gamma(X, L^k) < Ck^n, \tag{1.1}$$

where

- X is a complex manifold of dimension n.
- L is a holomorphic line bundle over X.
- L^k is the k^{th} tensor product.
- $\Gamma(X, L^k)$ is the space of global holomorphic sections $X \to L^k$.
- C is a constant, which depends on X and L but not on k.

We start in Section 2 with an explicit example and an easy calculation. One of the first explicit instances of this inequality occurs in the work of C. L.Siegel. In Section 3 we sketch his proof and application. In Section 4 we sketch important generalizations.

2. Background Material and An Example

We begin by recalling the relation between divisors and holomorphic line bundles. See [6] for details. A divisor in a complex manifold X is a locally-finite, formal sum

$$D = \sum_i a_i V_i$$

where $a_i \in Z$ and each V_i is an irreducible sub-variety of codimension one. We fix a sufficiently fine open covering

$$X = \cup U_\alpha$$

such that $V_i \cap U_\alpha$ has a single defining function $g_{i\alpha} \in \mathcal{O}(U_\alpha)$. Set

$$f_\alpha = \Pi_i g_{i\alpha}^{a_i}$$

and when $U_\alpha \cap U_\beta \neq \emptyset$ take

$$g_{\alpha\beta} = f_\alpha / f_\beta$$

as the transition functions from $U_\beta \times \mathbf{C}$ to $U_\alpha \times \mathbf{C}$ of a holomorphic line bundle on X. Denote this bundle by D. In particular, if we start with a global meromorphic function f, pass to the divisor that represents the zeros and poles of f, and then to the line bundle associated to the divisor, we end up with a trivial line bundle. Further, if a line bundle $[D]$ is trivial, then D is the divisor of a meromorphic function. At the end of this section, we will introduce another, and notrivial,

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line bundle associated with a meromorphic function. This construction will be important in Section 3.

Definition 1. *The hyperplane line bundle H on CP^n is the line bundle associated to any hyperplane divisor.*

That is, each hyperplane

$$H_\alpha = \{[z_1, \ldots, z_{n+1}] \mid \sum_{j=1}^{n+1} \alpha_j z_j = 0\} \subset CP^n$$

determines a line bundle and these bundles are all holomorphically isomorphic.

Here are two simple results. Together they show that the powers of H satisfy (1.1).

Lemma 2.1. *There is a bijection between the set of holomorphic sections of H^k and the set of homogeneous polynomials of degree k in $n+1$ variables.*

This can be seen directly by introducing the transition functions for $[H]$. An abstract proof is given in [6, page 165].

Lemma 2.2. *The dimension of the space of homogeneous polynomials of degree k in $n+1$ variables is*

$$\left(\frac{n+k}{n}\right)$$

and this quantity is of the order of Ck^n, where C depends only on n.

Proof. We count the monomials in $n+1$ variables of degree k. This is equivalent to a well-known counting problem: In how many ways may k identical balls be distributed into $n+1$ baskets? the answer is

$$\begin{pmatrix} n+k \\ n \end{pmatrix}.$$

Further, using Stirling's Estimate we see that

$$\begin{pmatrix} n+k \\ n \end{pmatrix} \sim \frac{\left(\frac{n+k}{e}\right)^{n+k} \sqrt{2\pi(n+k)}}{n! \left(\frac{k}{e}\right)^k \sqrt{2\pi k}} \tag{2.1}$$

$$< C_n k^n. \tag{2.2}$$

□

It is also possible to associate a line bundle to a meromorphic function is such a way that the function is a quotient of global sections of the bundle. Of course this bundle is usually non-trivial; it can be trivial only when the meromorphic function is in fact a global quotient of functions on X. Starting with a meromorphic function f, we may find a finite covering

$$X = \bigcup U_j$$

and holomorphic functions over U_j such that

$$f = \frac{p_j}{q_j} \text{ on } U_j, \text{ and } \frac{p_j}{q_j} = \frac{p_k}{q_k} \text{ on } U_j \cap U_k.$$

Let L be the line bundle with transition functions

$$g_{jk} = \frac{q_j}{q_k} \in \mathcal{O}^*(U_j \cap U_k).$$

Then the collection of local sections $\{p_j\}$ and $\{q_j\}$ satisfy

$$p_j = g_{jk} p_k$$

and

$$q_j = g_{jk} q_k$$

on $U_j \cap U_k$ and so both $p = \{p_j\}$ and $q = \{q_j\}$ define global sections of L. The global quotient p/q is our original meromorphic function f. This construction will be very useful in the next section.

3. A Theorem of Siegel

Here is one of the first general results of the type of (1.1). It is due to Siegel [8].

Theorem 3.1. *If X is a compact, complex manifold of dimension n and if L is a holomorphic line bundle over X, then there is some constant depending only on X and L such that*

$$\dim \Gamma(X, L^k) < Ck^n.$$

We outline Siegel's proof and then go over his beautiful application. In this section we are following [5] and [1].

We start with two open coverings of X, $\{W_a\}$ and $\{V_a\}$, with $W_a \subset V_a$ and each open set biholomorphic to a domain of holomorphy in $\mathbf{C}^n$. This implies that $L|_{V_a}$ is trivial. Given a global section

$$s : X \to L$$

we set

$$s_a = s|_{V_a}$$

and consider s_a to take values in $\mathbf{C}$. This allows us to define the norms

$$||s||_V = \max_a \sup_{z \in V_a} |s_a(z)|$$

and

$$||s||_W = \max_a \sup_{z \in W_a} |s_a(z)|.$$

Certainly

$$||s||_W \leq ||s||_V.$$

Lemma 3.1. *There exists a positive constant g, depending only on the transition functions for L with respect to the covering $\{W_a\}$, such that*

$$||s||_V \leq g||s||_W.$$

Proof. This is clear since each $p \in V_a$ is also in some W_b and so

$$s_a(p) = g_{ab} s_b(p).$$

Thus

$$|s_a(p)| \leq g||s||_W.$$

□

We now choose the open coverings in a special way. Let P_r denote the polydisc

$$P_r = \{z \in \mathbf{C}^n \mid ||z_j|| < r \text{ for } j = 1, \ldots, n\}.$$

We can find open coverings $\{W_a\}$ and $\{V_a\}$ and real numbers r_0 and r_1 with $0 < r_0 < r_1$ such that each W_a is biholomorphic to P_{r_0} and each V_a is biholomorphic to P_{r_1}. The Schwarz Lemma provides a proof of the following result.

Lemma 3.2. *If $F(z)$ is holomorphic on P_{r_1} and vanishes to order at heast h at the origin, then*

$$\sup_{P_{r_0}} |F| \leq \left(\frac{r_0}{r_1}\right)^h \sup_{P_{r_1}} |F|.$$

Thus if the section s vanishes to at least order h at each of the points of X corresponding to the origin in P_{r_1}, then

$$||s||_W \leq \left(\frac{r_0}{r_1}\right)^h ||s||_V \leq \left(\frac{r_0}{r_1}\right)^h g||s_W||. \tag{3.1}$$

Thus

$$h > \frac{\ln g}{\ln r_1 - \ln r_0} \tag{3.2}$$

implies that

$$s \equiv 0.$$

Let us summarize what has just been proved.

Lemma 3.3. *There exist a finite number of points $x_a \in X$, $a = 1, \ldots, N$ and an integer h such that the only section of L which vanishes to at least order h at each x_a is the zero section. Further, there exists a constant C depending only on a fixed covering of X so that h can be chosen to be any integer greater than $C \ln g$, where g depends only on bounds for the transition functions for L.*

We can now prove Theorem 3.1.

Proof. Let $\Gamma(X, L)$ be the complex vector space of holomorphic sections of L. Let $J(x_a)$ be the space of jets up to order h at x_a of holomorphic sections of L. Consider the map

$$\Gamma(X, L) \to \oplus_{a=1}^N J(x_a).$$

The Fundamental Lemma tells us that this map is injective. Since the dimension of each $J(x_a)$ is $\left(\begin{array}{c} n+h \\ h \end{array}\right)$,

we have

$$\dim \Gamma(X, L) \leq N \left(\begin{array}{c} n+h \\ h \end{array}\right).$$

We want to estimate the right hand side when h is large. We do this again using Stirling's formula.

So, as $h \to \infty$

$$\begin{aligned} \left(\begin{array}{c} n+h \\ h \end{array}\right) = \frac{(n+h)!}{h!n!} \quad &\sim \quad \frac{\sqrt{1+\frac{n}{h}}(n+h)^{n+h}}{e^n h^h n!} \\ &\sim \quad \frac{(n+h)^n}{n!} \\ &\sim \quad \frac{h^n}{n!}. \end{aligned}$$

Now we replace L by L^k. This means that the transition functions are replaced by their k^{th} powers and so g becomes g^k. Then h is replaced by $c'k$ for some c' depending on L and so

$$\dim \Gamma(X, L^k) < C \left(\begin{array}{c} n+c'k \\ c'k \end{array}\right) < C'k^n. \tag{3.3}$$

This completes the proof, based on [1], of Theorem 3.1. □

Siegel uses this theorem to give the first proof of a basic result in algebraic geometry.

Theorem 3.2. *The field of meromorphic functions on a compact, complex manifold X of dimension n is an algebraic extension of the field of rational functions in d variables, with $d \leq n$.*

Thus

$$K(X) \cong Q(t_1, \ldots, t_d, \theta),\ \theta \text{ algebraic in } t_1, \ldots, t_d.$$

Remark 1. *A compact complex manifold with $d = n$ is called a Moishezon manifold.*

It is not too hard to see that the theorem would follow if we could establish that any set of $n+1$ meromorphic functions on X is algebraically dependent.

Recall that meromorphic functions on a complex manifold X are analytically dependent if

$$df_1 \wedge \ldots \wedge df_m = 0$$

at each point of X at which the functions are all holomorphic. And they are algebraically dependent if there is a nontrivial polynomial P over C with

$$P(f_1, \ldots, f_m) = 0$$

at all such points. Applying d to this equation shows that algebraic dependence implies analytic dependence. Here is the converse.

Theorem 3.3. *If the meromorphic functions $f_1, \ldots, f_m$ on X are analytically dependent, then they are also algebraically dependent.*

This implies Theorem 3.2.

To prove this theorem we will need some minor modifications of Lemma 3.3. First we replace L^k by a line bundle of the form $L^k \otimes F^s$. The transition functions for $L^k \otimes F^s$ are of the form $g_{ij}^k f_{ij}^s$ and so are bounded by C^{k+s} for some C. So in the proof of Theorem 3.3 the inequality (3.2) is replaced by

$$h > \frac{-\ln C^{k+s}}{\ln q} \tag{3.4}$$

and inequality (3.3) is replaced by

$$\dim \Gamma(X, L^k \otimes F^s) < C \begin{pmatrix} n + c'(k+s) \\ c'(k+s) \end{pmatrix} < C'(k+s)^n. \tag{3.5}$$

Let E denote some line bundle and assume that local holomorphic coordinates $\zeta_1, \ldots, \zeta_n$ are specified in a neighborhoodof each x_a and that the only sections we consider are those that in a neighborhoodof x_a are holomorphic functions of only $\zeta_1, \ldots, \zeta_m$ for some $m \leq n$. Denote the space of such sections by $\Gamma_0(\overline{X}, E)$. We have restricted the set of sections so of course it still follows that a section vanishing to at least order h at each x_a must be identically zero. At each x_a there are $\begin{pmatrix} m+h \\ h \end{pmatrix}$ polynomials in m variables of degree less than or equal to h. So

$$dim\ \Gamma_0(\overline{X}, E) \leq N \begin{pmatrix} m+h \\ h \end{pmatrix}$$

and as before

$$dim\ \Gamma_0((\overline{X}, E^k) \leq C \left(\begin{array}{c} m + c'k \\ c'k \end{array} \right) \leq C'k^m.$$

And finally, we combine these two modifications.

$$dim\ \Gamma_0((\overline{X}, L^k \otimes F^s) \leq C'(k+s)^m. \tag{3.6}$$

We are now ready to prove Theorem 3.3. We change notation and start with analytically independent meromorphic functions $f_1, \ldots, f_m$ and a meromorphic function f with

$$df \wedge df_1 \wedge \ldots \wedge df_m = 0$$

at each point where this makes sense. We need to find a polynomial such that $P(f_1, \ldots f_m, f) = 0$ at each point where the functions are all holomorphic. Let L_j be the bundle associated to f_j and let F be the bundle associated to f. Then each f_j is a global quotient of sections of

$$L = L_1 \otimes L_2 \otimes \ldots \otimes L_m.$$

Further, $f_1^{k_1} \cdots f_m^{k_m} f^s$ is a global quotient of $L^k \otimes F^s$ where $k = k_1 + \ldots + k_m$.

We write

$$\begin{aligned} f_j &= \frac{s_j}{s_0} \qquad s_j, s_0 \text{ global sections of } L, \\ f &= \frac{\phi}{\psi} \qquad \phi, \psi \text{ global sections of } F. \end{aligned}$$

We fix some positive integers r and s. Let

$$\begin{aligned} W_0(r,s) = \{&\text{polynomials of degree at most } r \\ &\text{in each of } X_1, \ldots, X_m \\ &\text{and degree at most } s \text{ in } X_{m+1}\} \end{aligned}$$

We want to eliminate the denominators in our global quotients and also to work only with homogeneous polynomials. So let

$$W(r,s) = \{Q(\xi, \eta, X_1, \ldots, X_m, X_{m+1}) = \xi^{mr}\eta^s P(\frac{X_1}{\xi}, \ldots, \frac{X_m}{\xi}, \frac{X_{m+1}}{\eta}), P \in W_0(r,s)\}.$$

Thus $Q \in W(r,s)$ is homogeneous in the sense that

$$Q(a\xi, b\eta, aX_1, \ldots, aX_m, bX_{m+1}) = a^{mr}b^s Q(\xi, \eta, X_1, \ldots, X_m, X_{m+1}).$$

We may assume that at the points x_a in the proof of Theorem 1.1

$$df_1 \wedge \ldots \wedge df_m \neq 0.$$

So these functions define a partial set of local coordinates which we use to define $\Gamma_0(\overline{X}, L^{mr} \otimes F^s)$.

Next define

$$\Pi : W(r,s) \to \Gamma_0(\overline{X}, L^{mr} \otimes F^s)$$

by

$$\Pi Q = Q(s_0, \psi, s_1, \ldots, s_m, \phi).$$

It suffices to prove that Π is not injective. The modifications of Theorem 1.1 apply as long as (3.4) holds. Thus (3.6) holds with k replaced by mr:

$$dim\ \Gamma_0(\overline{X}, L^{mr} \otimes F^s) \leq C'(mr+s)^m.$$

It is easy to see that

$$dim\ W(r,s) = (r+1)^m(s+1)$$

and so we choose r and s such that

$$(r+1)^m(s+1) > C'(mr+s)^m.$$

Thus Π is not injective.

4. Generalizations

There are several natural directions in which to seek generalizations of Theorem 3.1. One is to replace the compact manifold by an open subset Ω of a complex manifold. Most analytic results about open subsets of a complex manifold are in the context of pseudoconvexity. But the appropriate replacement for compactness is pseudoconcavity.

Another direction is to replace holomorphic sections of L by $\overline{\partial}$-closed differential forms with values in L. But the space of such forms is infinite dimensional. (If s is a smooth section of L, then $\overline{\partial}s$ is a $\overline{\partial}$ -closed section of $L \otimes \Lambda^{(0,1)}(\Omega)$.) So one looks at the dimensions of the Dolbeault cohomology instead.

And finally, one could combine these generalizations and look for upper bounds for the cohomology of powers of a line bundle over an appropriate bounded domain.

4.1. **Pseudoconcavity.** Let $\Omega \subset\subset X$ be a domain in a complex manifold X of dimension n.

Definition 2. *A point $q \in b\Omega$ is an Andreotti pseudoconcave point if there exists a fundamental system of neighborhoods $\{U\}$ such that q is an interior point of each of the holomorphic hulls*

$$(\widehat{U \cap \Omega})_U = \{p \in U |\ |f(p)| \leq \sup_{z \in U\cap\Omega} |f(z)|, \forall f \in \mathcal{O}(U)\}.$$

A domain Ω is Andreotti pseudoconcave if each of its boundary points is.

Theorem 4.1. [1],[2] *If Ω is Andreotti pseudoconcave and relatively compact and if L is a holomorphic line bundle over $\overline{\Omega}$, then there is some constant depending only on Ω and L such that*

$$\dim \Gamma(\Omega, L^k) \leq Ck^n. \tag{4.1}$$

This is Andreotti's generalization of Siegel's Theorem.

So if the Levi form has at least one negative eigenvalue (4.1) holds. This condition can be relaxed to allow some degenerate Levi forms. A real hypersurface M in X is minimal (in the sense of Trepreau) at a point q if there does not exist the germ of complex hypersurface passing through q and contained in M. We say an open set is minimal at a boundary point if the boundary is minimal at that point.

Theorem 4.2. *If $b\Omega$ is minimal at all points and if the Levi form has no positive eigenvalues then* (4.1) *holds.*

For the proof, see [5].

4.2. **Demailly's Morse Inequalities.** Let M be a compact manifold and let f be a Morse function. Let C^j be the number of critical points of f of index j. The index of a function at a nondegenerate critical point is the number of negative eigenvalues of its Hessian at that point. Let b_j be the j^{th} Betti number, that is $b_j = dim\ H^j(M, R)$. The strong Morse inequalities assert that

$$C^j - C^{j-1} + \cdots \pm C^0 \geq b_j(M) - b_{j-1}(M) + \cdots \pm b_0(M).$$

In particular, for any

$$j \in \{0, \ldots, n = \dim M\},$$

we have the weak Morse inequalities

$$b_j(M) \leq C^j.$$

In a seminal paper [9], Witten outlined a new approach to the Morse inequalities by relating them to the asymptotic behavior of a "twisted" de Rham complex. In effect, the problem is localized at the critical points and then solved by a computation in local coordinates.

In [4], Demailly recognized that one could obtain Morse inequalities for the Dolbeault complex by replacing the study of the asymptotic behavior of the twisted de Rham complex by a high tensor product of a line bundle. See also [7]. This provides the extension of (1.1) to the dimensions of the cohomology groups. We state a simplified version of his result.

Theorem 4.3. *Let X be a compact complex manifold of dimension n and let L be a line bundle over X. Then*

$$\dim H^j(X, L^k) < Ck^n, \textit{ for } 0 \leq j \leq n$$

where C depends on X and L but not on k.

4.3. **Boundary behavior.** It is now natural to generalize to complex manifolds with boundary. Again, for ease of exposition, we present simplified results.

Theorem 4.4 (Berman [3])**.** *Let X be a complex manifold of dimension n with boundary and let L be a holomorphic line bundle over X. Assume that the Levy form is nondegenerate at each point of the boundary and that for some j, $0 \leq j \leq n$, the Levy form never has precisely j negative eigenvalues. Then*

$$dim H^j(X, L^k) < Ck^n$$

for some C that depends on X and L but not on k.

Finally, in the presence of pseudoconvexity we may allow some degeneracy of the Levy form at the boundary.

Theorem 4.5 (Fu-Jacobowitz [5])**.** *Let $\Omega \subset\subset X$ be a smooth pseudoconvex domain in a complex manifold and let L be a holomorphic line bundle over Ω that extends smoothly up to the boundary. Assume that there exists a bounded continuous function whose complex hessian is bounded from below by a positive constant in a neighborhood of $b\Omega$. Then for any $1 \leq j \leq n$,*

$$h^j(\Omega, L^k) \leq Ck^n$$

for some C that depends on Ω and L but not on k.

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