

TOTALLY REAL MAPPINGS AND INDEPENDENT MAPPINGS

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1. INTRODUCTION

We consider two classes of smooth maps $M^n \rightarrow \mathbf{C}^N$. All manifolds are assumed to be connected (unless otherwise mentioned) and to have countable topology.

Definition 1. A map $M^n \rightarrow \mathbf{C}^N$ is called a **totally real immersion (embedding)** if f is an immersion (embedding) and for $f_* : TM \rightarrow T\mathbf{C}^N$ we have

$$(1) \quad f_*(TM) \cap Jf_*(TM) = \{0\}.$$

Here we have identified \mathbf{C}^N with \mathbf{R}^{2N} together with the natural anti-involution J .

Definition 2. A map $f : M^n \rightarrow \mathbf{C}^N$ is called an **independent map** if

$$df_1(p) \wedge \cdots \wedge df_N(p) \neq 0$$

for $f = (f_1, \dots, f_N)$ and for all $p \in M$.

We are interested in the optimal value of N for all manifolds of dimension n . Sections 2 and 3 provide an exposition of [3], which gives some details not presented here. Section 4 discusses a special case where our two types of maps are related in a perhaps unexpected way.

2. EXISTENCE

Theorem 2.1. Any map $f : M^n \rightarrow \mathbf{C}^N$ may be approximated by a totally real embedding, provided $N \geq \lceil \frac{3n}{2} \rceil$ and $n \geq 2$.

Remark. For N and n satisfying these inequalities, any map of M into \mathbf{C}^N may be approximated by an embedding [6] and in particular any totally real immersion may be approximated by a totally real embedding.

Theorem 2.2. Any map $f : M^n \rightarrow \mathbf{C}^N$ may be approximated by an independent map, provided $N \leq \lceil \frac{n+1}{2} \rceil$.

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The proofs of these theorems depend on a well-known result from differential topology. Let $J^1(M, W)$ be the space of one-jets of maps from M to W . Denote the lift of any map

$$f : M \rightarrow W$$

by

$$j^1(f) : M \rightarrow J^1(M, W).$$

Theorem. *If $\Sigma \subset J^1(M, W)$ is stratified by locally closed submanifolds and $\dim M < \text{codim } \Sigma$ then there exists some $F : M \rightarrow W$ with $(j^1(F)M) \cap \Sigma = \emptyset$.*

The proof is straightforward, see for instance [1].

To prove Theorem 2.1 we take Σ to be given in local coordinates over an open set U by

$$\Sigma = \{(p, q, a^1, \dots, a^n) : p \in U, q \in \mathbf{R}^{2N}, \text{rank}(A, JA) < 2n\}$$

where

$$A = (a^1 \cdots a^n)$$

is a real $N \times n$ matrix and (A, JA) is the $N \times 2n$ matrix obtained by juxtaposition. To prove Theorem 2.2 we set $r = \lfloor \frac{n+1}{2} \rfloor$ and use

$$\Sigma = \{(p, q, \alpha^1, \dots, \alpha^n), \text{rank } A < r\}$$

where A is the complex $r \times n$ matrix

$$A = (\alpha^1 \cdots \alpha^n).$$

It is easy to verify that these are stratified subsets and that the given values of N lead to

$$\dim M < \text{codim } \Sigma.$$

3. OPTIMALITY

To explain our examples, we find necessary bundle-theoretic conditions for totally real immersions and for independent maps.

Lemma 3.1. (a) *If M has a totally real immersion into \mathbf{C}^N then there exists a bundle Q of rank $r = N - n$ such that*

$$(\mathbf{C} \otimes TM) \oplus Q \cong N\varepsilon.$$

(b) *If M has an independent map into \mathbf{C}^N then there exists a bundle B of rank $r = n - N$ such that*

$$\mathbf{C} \otimes TM \cong N\varepsilon \oplus B.$$

Here $N\varepsilon$ is the trivial complex vector bundle over M of rank N .

Remark. An application of the Gromov h-principle shows that these conditions are also sufficient. See a discussion of this in [4]. We will make use of the sufficiency below.

Proof of Lemma 3.1. (a) The condition (1) is equivalent to the fiber injectivity of

$$\phi_f : \mathbf{C} \otimes TM \rightarrow T^{1,0}(\mathbf{C}^N)$$

where $\phi_f(v)$ is defined, for $v \in \mathbf{C} \otimes TM$, by

$$\phi_f(v) = f_*(v) - iJf_*(v).$$

Thus if M has a totally real immersion into \mathbf{C}^N then

$$(\mathbf{C} \otimes TM) \oplus Q \cong N\varepsilon$$

where Q is the bundle in $T^{1,0}$ normal to $\phi_f(\mathbf{C} \otimes TM)$.

(b) The map

$$\psi_f : \mathbf{C} \otimes TM \rightarrow T^{1,0}$$

given by

$$\psi_f(v) = \sum df_j(v) \partial_{z_j}$$

is surjective on the fibers. So

$$\mathbf{C} \otimes TM \cong N\varepsilon \oplus B.$$

with $B = \ker \psi_f$. □

3.1. Totally real immersions. We need to find a manifold of dimension n that does not have a totally real immersion into \mathbf{C}^N for $N = \lceil \frac{3n}{2} \rceil - 1$. We provide four families of examples according to the residue of the dimension of M modulo 4. Let

$$M^{4k} = \mathbf{CP}^2 \times \cdots \times \mathbf{CP}^2 = (\mathbf{CP}^2)^{\times k}$$

be the product of k copies of the complex projective plane. The manifolds we use and the ensuing arguments are similar to those given by Forster [2], but we use orientable manifolds as far as possible.

Theorem 3.1.

- M^{4k} does not admit a totally real immersion into \mathbf{C}^N for $N = 6k - 1$.
- $M^{4k+1} = M^{4k} \times S^1$ does not admit a totally real immersion into \mathbf{C}^N for $N = 6k$.
- $M^{4k+2} = M^{4k} \times \mathbf{RP}^2$ does not admit a totally real immersion into \mathbf{C}^N for $N = 6k + 2$.
- $M^{4k+3} = M^{4k} \times \mathbf{RP}^2 \times S^1$ does not admit a totally real immersion into \mathbf{C}^N for $N = 6k + 3$.

Denote the total Chern class of a complex vector bundle B over M by

$$c(B) = 1 + c_1(B) + \cdots + c_k(B)$$

where $c_j(B) \in H^{2j}(M; \mathbf{Z})$ and $k = \min(\text{rank } B, \lceil \frac{\dim M}{2} \rceil)$. We have the following well-known result (see e.g. [5, Section 14]).

Lemma 3.2. *Let a denote the first Chern class of the hyperplane line bundle $\mathcal{O}(1)$ on \mathbf{CP}^2 . Then*

$$c(\mathbf{C} \otimes T\mathbf{CP}^2) = 1 - 3a^2.$$

We need to show that in the first two cases of Theorem 3.1 there is no bundle Q of rank $2k - 1$ and in the last two cases no bundle Q of rank $2k$ such that $(\mathbf{C} \otimes TM) \oplus Q$ is trivial. We shall show this for M^{4k+1} and M^{4k+3} . The other two cases, which are very similar to these, are done in [3]. So first we assume that there is some Q with

$$(\mathbf{C} \otimes TM^{4k+1}) \oplus Q \cong N\varepsilon$$

for some N and show that the rank of Q is at least $2k$.

Let a_1, \dots, a_k be the pull-backs of a to M under the corresponding projections to \mathbf{CP}^2 , so that $a_i^3 = 0$ for all i . We have

$$c(\mathbf{C} \otimes TM^{4k+1}) \cdot c(Q) = 1.$$

Thus $c(Q) = (1 + 3a_1^2) \cdots (1 + 3a_k^2)$. Since $a_1^2 \cdots a_k^2 \neq 0$, this implies that the rank of Q is at least $2k$.

Next we assume that there exists some Q with

$$(\mathbf{C} \otimes TM^{4k+3}) \oplus Q = N\varepsilon$$

for some N and show that the rank of Q is at least $2k + 1$. Let a_1, \dots, a_k be as before and let b_1 be the pull-back of the generator in $H^2(\mathbf{RP}^2; \mathbf{Z})$ given by the Chern class of the complexification of the tautological line bundle on \mathbf{RP}^2 . We have

$$c(\mathbf{C} \otimes TM^{4k+3}) \cdot c(Q) = 1$$

which now gives

$$c(Q) = (1 + 3a_1^2) \cdots (1 + 3a_k^2)(1 - b_1).$$

This implies that the rank of Q is at least $2k + 1$.

3.2. Independent maps. The same manifolds M^{4k+r} ($0 \leq r \leq 3$) show that Theorem 2.2 is also optimal.

Theorem 3.2.

- M^{4k} does not admit an independent map into \mathbf{C}^N for $N = 2k + 1$.
- $M^{4k+1} = M^{4k} \times S^1$ does not admit an independent map into \mathbf{C}^N for $N = 2k + 2$.
- $M^{4k+2} = M^{4k} \times \mathbf{RP}^2$ does not admit an independent map into \mathbf{C}^N for $N = 2k + 2$.
- $M^{4k+3} = M^{4k} \times \mathbf{RP}^2 \times S^1$ does not admit an independent map into \mathbf{C}^N for $N = 2k + 3$.

The proofs are similar to those of Theorem 3.1 and can be found in [3]. For instance, to show that M^{4k+1} does not admit an independent map into \mathbf{C}^N for $N = 2k + 2$ we start with

$$\mathbf{C} \otimes TM^{4k+1} \cong N\varepsilon \oplus B$$

for some N which gives us

$$c(B) = c(\mathbf{C} \otimes TM^{4k+1}) = (1 - 3a_1)^2 \cdots (1 - 3a_k^2).$$

So the rank of B is at least $2k$ and since $N + \text{rank } B = 4k + 1$, this leads to $N \leq 2k + 1$.

4. NEW RESULTS FOR FOUR-MANIFOLDS

The fact that the same set of examples demonstrates the optimality of both Theorems 2.1 and 2.2 suggests that for some class of manifolds the two conditions

$$(\mathbf{C} \otimes TM) \oplus Q \cong N\varepsilon.$$

and

$$\mathbf{C} \otimes TM \cong N\varepsilon \oplus B.$$

are related. As a first step in exploring this, we present a result for four-dimensional manifolds.

Theorem 4.1. *Let M be either an open or an orientable four-manifold. Then M has a totally real immersion into \mathbf{C}^5 if and only if M admits an independent map into \mathbf{C}^3 .*

This result is false (in both directions) for non-orientable 4-manifolds:

Theorem 4.2. \mathbf{RP}^4 admits an independent map into \mathbf{C}^3 , but no totally real immersion into \mathbf{C}^5 . Moreover, the connected sum of \mathbf{RP}^4 and $\mathbf{RP}^2 \times \mathbf{RP}^2$ admits a totally real immersion into \mathbf{C}^5 , but no independent map into \mathbf{C}^3 .

Proof of Theorem 4.1. The hypothesis on M implies that $H^4(M; \mathbf{Z})$ is either zero or is torsion-free. We will also use that $2c_1(\mathbf{C} \otimes TM) = 0$.

Let M have such a totally real immersion. So there exists a Q of rank 1 with

$$(2) \quad (\mathbf{C} \otimes TM) \oplus Q \cong 5\varepsilon$$

and we want to find a B (also of rank 1) such that

$$(3) \quad \mathbf{C} \otimes TM \cong 3\varepsilon \oplus B.$$

From

$$c((\mathbf{C} \otimes TM) \oplus Q) = 1$$

we derive

$$c_2(\mathbf{C} \otimes TM) = c_1(\mathbf{C} \otimes TM)^2.$$

Thus

$$2c_2(\mathbf{C} \otimes TM) = 0$$

which implies

$$c_2(\mathbf{C} \otimes TM) = 0.$$

Dimensional considerations imply that $\mathbf{C} \otimes TM \cong 2\varepsilon \oplus B'$, where B' is a rank two complex vector bundle. Note that $c_2(B') = c_2(\mathbf{C} \otimes TM)$ and so $c_2(B') = 0$. Since $c_2(B')$ coincides with the Euler characteristic of the underlying real oriented bundle, B' admits a global nowhere zero section (see [5, Theorem 12.5]). Thus

$$\mathbf{C} \otimes TM \cong 2\varepsilon \oplus B' \cong 2\varepsilon \oplus \varepsilon \oplus B = 3\varepsilon \oplus B$$

and so M admits an independent map into \mathbf{C}^3 .

Now, conversely, we start with

$$\mathbf{C} \otimes TM \cong 3\varepsilon \oplus B$$

for some B of rank 1 and prove that there exists some Q (also of rank 1) with

$$(\mathbf{C} \otimes TM) \oplus Q \cong 5\varepsilon.$$

We see that

$$c(\mathbf{C} \otimes TM) = c(B).$$

This yields

$$c_1(\mathbf{C} \otimes TM) = c_1(B),$$

so that

$$2c_1(B) = 0$$

which implies that

$$(c_1(B))^2 = 0$$

by the hypothesis on M . From Theorem 2.1 and Lemma 3.1(a) we have

$$(\mathbf{C} \otimes TM) \oplus Q' \cong 6\varepsilon$$

for some Q' of rank 2. It remains to show that $Q' \cong Q \oplus \varepsilon$, which will follow from $c_2(Q') = 0$; the latter equation holds since

$$c(Q') = c(B)^{-1} = (1 + c_1(B))^{-1} = 1 + c_1(B).$$

□

Remark. We further observe that when (2) and (3) hold, it follows that $Q \cong B$ and that these line bundles have trivial square since their first Chern classes have order 2. Moreover, from the Corollary below we see that when M is orientable these line bundles are in fact trivial, so that M admits a totally real immersion into \mathbf{C}^4 .

We use the following lemma in the proof of Theorem 4.2.

Lemma 4.1. (a) M^4 has a totally real immersion into \mathbf{C}^5 if and only if the dual Pontryagin class $\bar{p}_1(M)$ is zero.
 (b) M^4 admits an independent map into \mathbf{C}^3 if and only if the Pontryagin class $p_1(M)$ is zero.

Proof. Assume that M^4 has a totally real immersion into \mathbf{C}^5 , so there exists a line bundle Q such that (2) holds. The dual Pontryagin class is defined by taking any immersion of M into some \mathbf{R}^m and setting $\bar{p}_1(M) = -c_2(\mathbf{C} \otimes N)$, where N is the normal bundle of the immersion. Thus $TM \oplus N$ is trivial. Since

$$(\mathbf{C} \otimes TM) \oplus (\mathbf{C} \otimes N)$$

is also trivial, we see that $c_2(\mathbf{C} \otimes N) = c_2(Q) = 0$. Thus $\bar{p}_1(M) = 0$.

Conversely, assume $\bar{p}_1(M) = 0$. As before, we have $(\mathbf{C} \otimes TM) \oplus Q' \cong 6\varepsilon$ with Q' of rank 2. We have $\bar{p}_1(M) = -c_2(Q')$, so $Q' \cong Q \oplus \varepsilon$ and we then can conclude that $(\mathbf{C} \otimes TM) \oplus Q$ is trivial.

On the other hand, the first Pontryagin class of M is equal, up to sign, to $c_2(\mathbf{C} \otimes TM)$. Thus if

$$\mathbf{C} \otimes TM = 3\varepsilon \oplus B,$$

with B of rank 1, we have $p_1(M) = 0$.

Finally, suppose that $p_1(M) = 0$, i.e. $c_2(\mathbf{C} \otimes TM) = 0$. We can write $\mathbf{C} \otimes TM \cong 2\varepsilon \oplus B'$, where $\text{rank } B' = 2$. So $c_2(B') = 0$, which implies that $B' \cong \varepsilon \oplus B$, yielding $\mathbf{C} \otimes TM \cong 3\varepsilon \oplus B$, as required. □

Theorem 4.2 is now an immediate consequence of the following lemma.

Lemma 4.2. $p_1(\mathbf{RP}^4) = 0$ and $\bar{p}_1(\mathbf{RP}^4) \neq 0$, while the opposite is true for the connected sum of \mathbf{RP}^4 and $\mathbf{RP}^2 \times \mathbf{RP}^2$.

Proof. Note first that for a closed connected non-orientable 4-manifold M the coefficient homomorphism $H^4(M; \mathbf{Z}) \rightarrow H^4(M; \mathbf{Z}_2)$ induced by reduction mod 2 is an isomorphism, as follows from the long exact sequence induced by the short exact sequence

$$0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}_2 \rightarrow 0$$

of coefficient groups, making use of the fact that $H^4(M; \mathbf{Z}) \cong \mathbf{Z}_2$. It is well known that for any real vector bundle over M this coefficient homomorphism sends p_1 to $(w_2)^2$ (see e.g. [5, Problem 15-A]). So $p_1(M) = 0$ if and only if $((w_2(M))^2 = 0$; and $\bar{p}_1(M) = 0$ if and only if $((\bar{w}_2(M))^2 = 0$.

For \mathbf{RP}^4 we have $w(\mathbf{RP}^4) = (1+x)^5 = 1+x+x^4$, where x denotes the generator in 1-dimensional cohomology. So $w_2(\mathbf{RP}^4) = 0$ and therefore also $p_1(\mathbf{RP}^4) = 0$. In addition, $\bar{w}(\mathbf{RP}^4) = 1+x+x^2+x^3$, so $\bar{w}_2(\mathbf{RP}^4) = x^2$ and it follows that $\bar{p}_1(\mathbf{RP}^4) \neq 0$.

Now let $M = \mathbf{RP}^4 \# (\mathbf{RP}^2 \times \mathbf{RP}^2)$. As this manifold is cobordant to the disjoint union of its two “summands”, its Stiefel-Whitney and dual Stiefel-Whitney numbers are the sums of the corresponding characteristic numbers of its summands. We have determined these characteristic numbers for \mathbf{RP}^4 , and need only add to them the corresponding characteristic numbers for $\mathbf{RP}^2 \times \mathbf{RP}^2$. One easily computes that for both $(w_2)^2$ and $(\bar{w}_2)^2$ the characteristic numbers of $\mathbf{RP}^2 \times \mathbf{RP}^2$ are nonzero. It follows that $p_1(M) \neq 0$, while $\bar{p}_1(M) = 0$. \square

In the case of an orientable four-manifold, we can obtain the following more precise result.

Corollary 4.1. *Let M be an orientable 4-manifold. Then $p_1(M)$ is zero if and only if $\bar{p}_1(M)$ is zero, and these conditions are equivalent to the existence of a totally real immersion of M into \mathbf{C}^4 . When these conditions fail, M admits no totally real immersion into \mathbf{C}^5 , nor an independent map into \mathbf{C}^3 .*

Proof. Let M be an orientable 4-manifold. Suppose that M admits a totally real immersion into \mathbf{C}^5 . In this case, we know that $(\mathbf{C} \otimes TM) \oplus Q$ is trivial for a complex line bundle Q . Hence we have $c_1(\mathbf{C} \otimes TM) = -c_1(Q)$.

Since M is orientable, the top exterior power of TM is a trivial real line bundle, hence the top exterior power of $\mathbf{C} \otimes TM$ is a trivial complex line bundle, and so (e.g., by using the splitting principle) $c_1(\mathbf{C} \otimes TM) = 0$. Hence Q is a trivial line bundle.

It follows that $\mathbf{C} \otimes TM$ is stably trivial, and therefore is trivial for dimensional reasons. In turn, this implies that M admits a totally real immersion into \mathbf{C}^4 .

Similarly, if an orientable 4-manifold M admits an independent map into \mathbf{C}^3 , then in fact M admits a totally real immersion into \mathbf{C}^4 . \square

Remark. We provide a summary of the results in this section for a four-dimensional manifold M , in terms of the following list of conditions that the manifold may satisfy:

- (1) M admits a totally real immersion into \mathbf{C}^5 .
- (2) M admits a totally real immersion into \mathbf{C}^4 .
- (3) M admits an independent map into \mathbf{C}^3 .
- (4) M admits an independent map into \mathbf{C}^4 .
- (5) $\mathbf{C} \otimes TM$ is trivial.
- (6) The first dual Pontryagin class of M vanishes.
- (7) The first Pontryagin class of M vanishes.

Then:

- (a) Conditions (2), (4), and (5) are equivalent for all 4-manifolds, and plainly imply the remaining conditions.
- (b) Conditions (1) and (6) are equivalent for all 4-manifolds. The same holds for Conditions (3) and (7).
- (c) Conditions (1), (3), (6), and (7) are all satisfied if M is open.
- (d) All seven conditions are equivalent if M is orientable.
- (e) By Theorem 4.2, conditions (1) and (3) are not equivalent for compact non-orientable manifolds; indeed, neither implies the other.
- (f) The conditions (1), (3), (6), and (7) are satisfied by the non-orientable manifolds $\mathbf{RP}^2 \times \mathbf{R}^2$ and $\mathbf{RP}^2 \times S^2$, but these manifolds do not satisfy the conditions (2), (4), and (5), since in both case the first Chern class of the complexified tangent bundle is nonzero.

It seems unlikely that such complete results can be obtained for manifolds of larger dimension.

5. A GEOMETRIC APPROACH TO THEOREM 4.1

Here is an alternative proof that the equation (3),

$$\mathbf{C} \otimes TM = 3\varepsilon \oplus B,$$

implies, for orientable M^4 , that there exists some Q satisfying equation (2),

$$(\mathbf{C} \otimes TM) \oplus Q = 5\varepsilon.$$

The zero set of a generic section σ_1 of the line bundle B is a (possibly not connected) 2-dimensional orientable submanifold $Y \subset M$. In the usual way we have $2c_1(B) = 0$ and so this is also true for the restriction of B to Y . But since Y is orientable we may conclude that also $c_1(B|_Y) = 0$, which implies that $B|_Y$ is trivial. Let σ_2 be a nonzero section of $B|_Y$ and extend σ_2 smoothly to a section over M . These two sections provide a fiber-surjective map $M \times \mathbf{C}^2 \rightarrow B$. In light of (3), we then have a fiber-surjective map

$$M \times \mathbf{C}^5 \rightarrow \mathbf{C} \otimes TM$$

and this leads to (2).

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