THE $\overline{\partial}$-COHOMOLOGY GROUPS, HOLOMORPHIC MORSE INEQUALITIES, AND THE FINITE TYPE CONDITIONS

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Dedicated to Professor J. J. Kohn on the occasion of his 75^{th} *birthday.*

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1. Introduction

A classical theorem of Siegel [Sie55] says that the dimension of global holomorphic sections of the k^{th} tensor power E^k of a holomorphic line bundle E over a compact complex manifold X of dimension n grows at a rate of at most k^n as k tends to infinity. This theorem has important implications in complex algebraic geometry. For example, Siegel proved that, as a consequence, the algebraic degree of X (*i.e.*, the transcendence degree of the field of meromorphic functions on X) is less than or equal to n. (We refer the reader to [And73] for an exposition for relevant results.)

The classical Morse inequalities on compact Riemannian manifolds, relating the Betti numbers to the Morse indices, showcase interplays among analysis, geometry, and topology (*e.g.*, [Mil63]). In an influential paper [W82], Witten provided an analytic approach to the Morse inequalities. Instead of studying the deRham complex directly, Witten used the twisted deRham complex $d_{tf} = e^{-tf} d e^{tf}$, where d is the exterior differential operator and f the Morse function. The Morse inequalities then follows from spectral analysis of the twisted Laplace-Beltrami operator by letting $t \to \infty$. (See, for example, [HS85, Bis86, Z01, HN05] and references therein for detailed expositions of Witten's approach.)

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Asymptotic Morse inequalities for compact complex manifolds were established by Demailly ([De85]; see also [De89]). Demailly's Morse inequalities were inspired in part by Siu's solution [Siu85] to the Grauert-Riemenschneider conjecture [GR70] which states that a compact complex manifold with a semi-positive holomorphic line bundle that is positive on a dense subset is necessarily Moishezon (*i.e.*, its algebraic degree is the same as the dimension of the manifold). It is noteworthy that whereas the underpinning of Witten's approach is a semi-classical analysis of Schrödinger operators without magnetic fields, Demailly's holomorphic Morse inequality is connected to Schrödinger operators with strong magnetic fields. (Interestingly, a related phenomenon also occurs in compactness in the $\overline{\partial}$-Neumann problem for Hartogs domains in $\mathbb{C}^2$ (see [FS02, CF05]): Whereas Catlin's property (P) can be phrased in terms of semi-classical limits of non-magnetic Schrödinger operators, compactness of the $\overline{\partial}$-Neumann operator reduces to Schrödinger operators with degenerated magnetic fields.) More recently, Berman established a local version of holomorphic Morse inequalities on compact complex manifolds [Ber04] and generalized Demailly's holomorphic Morse inequalities to complex manifolds with non-degenerated boundaries [Ber05].

Here we study spectral behavior of the complex Laplacian for a relatively compact domain in a complex manifold whose boundary has a degenerated Levi form. In particular, we are interested in Siegel type estimates for such a domain. Let $\Omega \subset\subset X$ be a domain with smooth boundary in a complex manifold of dimension n. Let E be a holomorphic line bundle over Ω that extends smoothly to $b\Omega$. Let $h^q(\Omega, E)$ be the dimension of the Dolbeault cohomology group on Ω for $(0, q)$-forms with values in E. Let $\widetilde{h}^q(\Omega, E)$ be the dimension of the corresponding L^2-cohomology group for the $\overline{\partial}$-operator (see Section 2 for the precise definitions). It was proved by Hörmander that when $b\Omega$ satisfies conditions a_q and a_{q+1}[1], then these two cohomology groups are isomorphic. Furthermore, there exists a defining function r of Ω and a constant c_0, independent of E, such that these cohomology groups are isomorphic to their counterparts on $\Omega_c = \{z \in \Omega \mid r(z) < -c\}$ for all $c \in (0, c_0)$. Our first result is an observation that combining Hörmander's theorems [H65] with a theorem of Diederich and Fornæss [DF77] yields the following:

Theorem 1.1. *Let Ω be pseudoconvex. Assume that there exists a neighborhood U of $b\Omega$ and a bounded continuous function whose complex hessian is bounded from below by a positive constant on $U \cap \Omega$. Then $h^q(\Omega, E) = \widetilde{h}^q(\Omega, E)$ for all $1 \leq q \leq n$. Furthermore, there exists a defining function of Ω and a constant $c_0 > 0$, independent of E, such that $b\Omega_c$ is strictly pseudoconvex and $h^q(\Omega, E) = \widetilde{h}^q(\Omega_c, E)$ for all $c \in (0, c_0)$.*

It is well known that a smooth pseudoconvex domain of finite type in a complex surface satisfies the assumption in the above theorem ([Ca89, FoS89]; see Section 3). Together with Berman's result, one then obtains a holomorphic Morse inequality for such pseudoconvex domains. In particular, $h^q(\Omega, E^k) \leq Ck^n$, $1 \leq q \leq n$, for some constant $C > 0$. For pseudoconcave domains, we have

Theorem 1.2. *Assume that Ω is pseudoconcave and $b\Omega$ does not contain the germ of any complex hypersurface. Then $h^0(\Omega, E^k) \leq Ck^n$ for some constant $C > 0$.*

Notice that the above theorems show that $h^q(\Omega, E^k)$ is insensitive to the order of degeneracy of the Levi form of the boundary. This is related to the fact that the dimensions of cohomology groups (equivalently, the multiplicity of the zero eigenvalues of the $\overline{\partial}$-Neumann

[1] Recall that the boundary $b\Omega$ satisfies condition a_q if the Levi form of its defining function has either at least $q + 1$ negative eigenvalues or at least $n - q$ positive eigenvalues at every boundary point

Laplacian) alone, even though can determine pseudoconvexity (see [Fu05] for a discussion on related results), are not sufficient to detect other geometric features, such as the finite type conditions, of the boundary. For this, we need to consider higher eigenvalues. Let $N_k(\lambda)$ be the number of eigenvalues that are less than or equal to λ of the $\overline{\partial}$-Neumann Laplacian on Ω for $(0,1)$-forms with values in E^k. The following is the main theorem of the paper:

Theorem 1.3. *Let $\Omega \subset\subset X$ be pseudoconvex domain with smooth boundary in a complex surface X. Let E be a holomorphic line bundle over Ω that extends smoothly to $b\Omega$. Then for any $C > 0$, $N_k(Ck)$ has at most polynomial growth as $k \to \infty$ if and only if $b\Omega$ is of finite type.*

The proof of the above theorem is a modification of the arguments in [Fu05b]; we need only to establish here that effects of the curvatures of the metrics on the complex surface X and the line bundle E^k are negligible.

Our paper is organized as follows. In Section 2, we review definitions and notations, and provide necessary backgrounds. we prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4. The rest of the paper is devoted to the proof of Theorem 1.3. For the reader's convenience, we have made an effort to have the paper self-contained. This results in including previously known arguments in the paper.

2. Preliminaries

2.1. **The $\overline{\partial}$-Neumann Laplacian.** We first review the well-known setup for the $\overline{\partial}$-Neumann Laplacian on complex manifolds. (We refer the reader to [H65, FoK72, CS99] for extensive treatises of the $\overline{\partial}$-Neumann problem and to [De] for L^2-theory of the $\overline{\partial}$-operator on complex manifolds.) Let X be a complex manifold of dimension n. Let E be a holomorphic vector bundle of rank r over X. Let

$$C^\infty_{0,q}(X,E) = C^\infty(X, \Lambda^{0,q}T^*X \otimes E), \quad C^\infty_{0,*}(X,E) = \oplus_{q=0}^n C^\infty_{0,q}(X,E).$$

Let $\theta\colon E|_U \to U \times \mathbb{C}^r$ be a local holomorphic trivialization of E over U. Let ε_j, $1 \le j \le r$, be the standard basis for $\mathbb{C}^r$ and let $e_j = \theta^{-1}(x, \varepsilon_j)$, $1 \le j \le r$, be the corresponding local holomorphic frame of $E|_U$. For any $s \in C^\infty_{0,q}(U,E)$, we make the identification

$$s = \sum_{j=1}^r s_j \otimes e_j \simeq_\theta (s_1, \ldots, s_r),$$

where $s_j \in C^\infty_{0,q}(U,\mathbb{C})$, $1 \le j \le r$. Then the *canonical* $(0,1)$-*connection* is by definition given by

$$\overline{\partial}_q s \simeq_\theta (\overline{\partial}_q s_1, \ldots, \overline{\partial}_q s_r) \in C^\infty_{0,q+1}(M,E),$$

where $\overline{\partial}_q$ is the projection of the exterior differential operator onto $C^\infty(X, \Lambda^{0,q}T^*X)$.

Now assume that X is equipped with a hermitian metric h, given in local holomorphic coordinates $(z_1, \ldots, z_n)$ by

$$h = \sum_{j,k=1}^n h_{jk} dz_j \otimes d\bar{z}_k,$$

where (h_{jk}) is a positive hermitian matrix. Let Ω be a domain in X. Let E be a holomorphic vector bundle over Ω that extends smoothly to $b\Omega$. Assume that E is equipped with a smoothly varying hermitian fiber metric g, given in a local holomorphic frame $\{e_1, \ldots, e_r\}$

by $g_{jk} = \langle e_j, e_k\rangle$. For $u, v \in \mathbb{C}^\infty_{0,*}(X, E)$, let $\langle u, v\rangle$ be the point-wise inner product of u and v, and let

$$\langle\langle u, v\rangle\rangle_\Omega = \int_\Omega \langle u, v\rangle dV$$

be the inner product of u and v over Ω. Let $L^2_{0,q}(\Omega, E)$ be the completion of the restriction of $C^\infty_{0,q}(X, E)$ to Ω with respect to the inner product $\langle\langle\cdot,\cdot\rangle\rangle_\Omega$. We also use $\overline{\partial}_q$ to denote the closure of $\overline{\partial}_q$ on $L^2_{0,q}(\Omega, E)$. Thus $\overline{\partial}_q\colon L^2_{0,q}(\Omega, E) \to L^2_{0,q+1}(\Omega, E)$ is a densely defined, closed operator on Hilbert spaces. Let $\overline{\partial}^*_q$ be its Hilbert space adjoint.

Let

$$Q^E_{\Omega,q}(u, v) = \langle\langle\overline{\partial}_q u, \overline{\partial}_q v\rangle\rangle_\Omega + \langle\langle\overline{\partial}^*_{q-1} u, \overline{\partial}^*_{q-1} v\rangle\rangle_\Omega$$

be the sesquilinear form on $L^2_{0,q}(\Omega, E)$ with domain $\mathcal{D}\,(Q^E_{\Omega,q}) = \mathcal{D}\,(\overline{\partial}_q)\cap\mathcal{D}\,(\overline{\partial}^*_{q-1})$. Then $Q^E_{\Omega,q}$ is densely defined and closed. It then follows from general operator theory (see [Dav95]) that $Q^E_{\Omega,q}$ uniquely determines a densely defined self-adjoint operator $\Box^E_{\Omega,q}\colon L^2_{0,q}(\Omega, E) \to L^2_{0,q}(\Omega, E)$ such that

$$Q^E_{\Omega,q}(u, v) = \langle\langle u, \Box^E_{\Omega,q} v\rangle\rangle, \qquad \text{for } u \in \mathcal{D}\,(Q^E_{\Omega,q}), v \in \mathcal{D}\,(\Box^E_{\Omega,q}),$$

and $\mathcal{D}\,((\Box^E_{\Omega,q})^{1/2}) = \mathcal{D}\,(Q^E_{\Omega,q})$. This operator $\Box^E_{\Omega,q}$ is called *the $\overline{\partial}$-Neumann Laplacian* on $L^2_{0,q}(\Omega, E)$. It is an elliptic operator with non-coercive boundary conditions. It follows from the work of Kohn [Ko63, Ko64, Ko72] and Catlin [Ca83, Ca87] that it is subelliptic when Ω is a relatively compact and smoothly bounded pseudoconvex domain of finite type in the sense of D'Angelo [D82, D93]: There exists an $\varepsilon \in (0, 1/2]$ such that

$$\|u\|^2_\varepsilon \le C(Q^E_{\Omega,q}(u, u) + \|u\|^2)$$

for all $u \in \mathcal{D}\,(Q^E_{\Omega,q})$, where $\|\cdot\|_\varepsilon$ denotes the L^2-Sobolev norm of order ε on Ω.

2.2. **The Dolbeault and L^2-cohomology groups.** The Dolbeault and the L^2-cohomology groups on Ω with values in E are given respectively by

$$H_{0,q}(\Omega, E) = \frac{\{f \in C^\infty_{0,q}(\Omega, E) \mid \overline{\partial}_q f = 0\}}{\{\overline{\partial}_{q-1} g \mid g \in C^\infty_{0,q-1}(\Omega, E)\}}$$

and

$$\widetilde{H}_{0,q}(\Omega, E) = \frac{\{f \in L^2_{0,q}(\Omega, E) \mid \overline{\partial}_q f = 0\}}{\{\overline{\partial}_{q-1} g \mid g \in \mathcal{D}\,(\overline{\partial}_{q-1})\}}.$$

It follows from general operator theory that $\widetilde{H}_{0,q}(\Omega, E)$ is isomorphic to $\mathcal{N}\,(\Box^E_{\Omega,q})$, the null space of $\Box^E_{\Omega,q}$, when $\overline{\partial}_{q-1}$ has closed range. Furthermore, $\widetilde{H}_{0,q}(\Omega, E)$ is finite dimensional when $\Box^E_{\Omega,q}$ has compact resolvent, in particular, when it is subelliptic. It was shown by Hörmander [H65] that when $b\Omega$ satisfies conditions a_q and a_{q+1}, the L^2-cohomology group $\widetilde{H}_{0,q}(\Omega, E)$ is isomorphic to the Dolbeault cohomology group $H_{0,q}(\Omega, E)$.

2.3. **The spectral kernel.** Assume that $\Box^E_{\Omega,q}$ has compact resolvent. Let $\{\lambda^q_j; j = 1, 2, \ldots\}$ be its eigenvalues, arranged in increasing order and repeated according to multiplicity. Let φ^q_j be the corresponding normalized eigenforms. The *spectral resolution* $\mathbf{E}^E_{\Omega,q}(\lambda)\colon L^2_{0,q}(\Omega, E) \to L^2_{0,q}(\Omega, E)$ of $\Box^q_\Omega$ is given by

$$\mathbf{E}^E_{\Omega,q}(\lambda)u = \sum_{\lambda_j\le\lambda} \langle\langle u, \varphi^q_j\rangle\rangle\varphi^q_j.$$

Let $e^E_{\Omega,q}(\lambda; z, w)$ be the spectral kernel, i.e., the Schwarz kernel of $\mathbf{E}^E_{\Omega,q}(\lambda)$. Then

$$\operatorname{tr} e^E_{\Omega,q}(\lambda; z, z) = \sum_{\lambda_j \le \lambda} |\varphi^q_j(z)|^2.$$

Let

$$S^E_{\Omega,q}(\lambda; z) = \sup\{|\varphi(z)|^2 \mid \varphi \in \mathbf{E}^E_{\Omega,q}(\lambda)(L^2_{0,q}(\Omega, E)), \|\varphi\| = 1\}.$$

It is easy to see that

$$S^E_{\Omega,q}(\lambda; z) \le \operatorname{tr} e^E_{\Omega,q}(\lambda; z, z) \le \frac{n!}{q!(n-q)!} S^E_{\Omega,q}(\lambda; z). \tag{2.1}$$

(see, e.g., Lemma 2.1 in [Ber04].)

3. Isomorphism between the Dolbeault and L^2 cohomology groups

The following proposition is a simple variation of a result due to Diederich and Fornæss [DF77]. We provide a proof, following Sibony ([Sib87, Sib89]), for completeness.

Proposition 3.1. *Let $\Omega \subset\subset X$ be pseudoconvex with smooth boundary. Assume that there exists a neighborhood U of $b\Omega$ and a bounded continuous function whose complex hessian is bounded below by a positive constant. Then there exists an $\eta \in (0, 1)$, a smooth defining function $\tilde{r}$ of Ω, and a constant $C > 0$ such that $\rho = -(-\tilde{r})^\eta$ satisfies*

$$L_\rho(z, \xi) = \partial\bar{\partial}\rho(\xi, \bar{\xi}) \ge C|\rho(z)||\xi|^2_h \tag{3.1}$$

for all $z \in \Omega \cap U$ and $\xi \in T^{1,0}_z(X)$.

Proof. Let r be a defining function of Ω. Let $V \subset\subset U$ be a tubular neighborhood of $b\Omega$ such that the projection from $z \in V$ onto the closest point $\pi(z) \in b\Omega$ is well-defined and smooth. Shrinking V if necessary, we may assume that both z and $\pi(z)$ are contained in the same coordinate patch. By decomposing $\xi \in T^{1,0}_z(X)$ into complex tangential and normal components, we then obtain that

$$L_r(z, \xi) \ge -C_1(|r(z)||\xi|^2 + |\xi||\langle \partial r(z), \xi\rangle|) \tag{3.2}$$

for some constant $C_1 > 0$.

Write $\rho(z) = \varphi(r(z))e^{f(z)}$ where φ is a smooth function on $(-\infty, 0)$ and $f(z)$ on U. Then it follows from direct calculations that

$$L_\rho(z, \xi) = e^f \left(\varphi' L_r(z, \xi) + \varphi''|\langle \partial r, \xi\rangle|^2 + \varphi L_f(z, \xi) + \varphi|\langle \partial f, \xi\rangle|^2 + 2\varphi' \operatorname{Re} \langle \partial r, \xi\rangle\overline{\langle \partial f, \xi\rangle}\right). \tag{3.3}$$

Let $\varphi(t) = -(-t)^\eta$. Let $A > 0$ be a constant to be determined. Using the inequalities

$$2|\xi||\langle \partial r, \xi\rangle| \le \frac{A|r|}{\eta}|\xi|^2 + \frac{\eta}{A|r|}|\langle \partial r, \xi\rangle|^2$$

and

$$2|\langle \partial r, \xi\rangle\overline{\langle \partial f, \xi\rangle}| \le \frac{A|r|}{\eta}|\langle \partial f, \xi\rangle|^2 + \frac{\eta}{A|r|}|\langle \partial r, \xi\rangle|^2,$$

we then obtain from (3.2) and (3.3) that

$$\begin{aligned} L_\rho(z, \xi) \ge |\rho|\big(& -C_1(\eta + \frac{1}{2}A)|\xi|^2 + \frac{\eta}{|r|^2}(1 - \eta - \frac{\eta}{A} - \frac{C_1\eta}{2A})|\langle \partial r, \xi\rangle|^2 \\ & - L_f(z, \xi) - (1 + A)|\langle \partial f, \xi\rangle|^2\big). \end{aligned} \tag{3.4}$$

By Richberg's theorem, we may assume that there exists a bounded $g \in C^\infty(U \cap \Omega)$ such that $L_g(z, \xi) \geq C|\xi|^2$ for some $C > 0$. By rescaling g, we may further assume that $-2 \leq g \leq -1$. Let $f = -e^g$. Then on V,

$$-L_f(z,\xi) - (1+A)|\langle \partial f, \xi\rangle|^2 = e^g(L_g(z,\xi) + (1-(1+A)e^g)|\langle \partial g, \xi\rangle|^2). \tag{3.5}$$

Now choosing A and then η sufficiently small, we then obtain (3.1) on $V \cap \Omega$. We extend ρ to a strictly plurisubharmonic function on $U \cap \Omega$ by letting $\widetilde{\rho}(z) = \theta(\rho) + \delta\chi(z)g(z)$ where θ is a smooth convex increasing function such that $\theta(t) = t$ for $|t| \leq \varepsilon$ and is constant when $t < -2\varepsilon$ for sufficiently small $\varepsilon > 0$, $\chi(z) \in C_c^\infty(\Omega)$ is identically 1 on a neighborhood of $\Omega \setminus V$, and δ is sufficiently small. The desirable defining function is obtained by letting $\widetilde{r} = -(-\widetilde{\rho})^{1/\eta}$. □

Let $c_0 > 0$ be any sufficiently small constant such that $\{\rho = -c_0^\eta\} \subset U \cap \Omega$ (following the notations of Proposition 3.1). Theorem 1.1 is then a consequence of the combination of the above proposition and the results in Chapter III in [H65][2]. More specifically, that $h^q(\Omega, E) = \widetilde{h}^q(\Omega_c, E)$, $1 \leq q \leq n$, for any $c \in (0, c_0)$ follows from Theorem 3.4.9 in [H65]. The proof of $\widetilde{h}^q(\Omega, E) = \widetilde{h}^q(\Omega_c, E)$ also follows along the line of the proof of Theorem 3.4.9. We provide details as follows: Since Ω_c is strictly pseudoconvex, $\widetilde{H}_{0,q}(\Omega_c, E)$ is finite dimensional. To prove that the restriction map $\widetilde{H}_{0,q}(\Omega, E) \to \widetilde{H}_{0,q}(\Omega_c, E)$ is onto, one needs only to show that the restriction of the nullspace $\mathcal{N}(\overline{\partial}_q, \Omega)$ to Ω_c is dense in $\mathcal{N}(\overline{\partial}_q, \Omega_c)$. Let $\{c_j\}_{j=1}^\infty$ be an decreasing sequence of positive numbers approaching 0 with $c_1 = c$. Let $f = f_1 \in \mathcal{N}(\overline{\partial}_q, \Omega_{c_1})$ and let $\varepsilon > 0$. By applying Theorem 3.4.7 in [H65] inductively, we obtain $f_j \in \mathcal{N}(\overline{\partial}_q, \Omega_{c_j})$ such that

$$\|f_j - f_{j+1}\|_{\Omega_{c_j}} \leq \frac{\varepsilon}{2^j}.$$

It follows that there exists some $g \in \mathcal{N}(\overline{\partial}_q, \Omega)$ such that for any k, $\|f_j - g\|_{\Omega_{c_k}} \to 0$ as $j \to \infty$, and $\|f - g\|_{\Omega_c} \leq \varepsilon$. Hence the restriction map is surjective.

To prove the injectivity of the restriction map, it suffices to prove that for any $f \in L^2_{0,q}(\Omega, E)$ such that $\overline{\partial}_q f = 0$ on Ω and $f = \overline{\partial}_{q-1} u$ on Ω_c for some $u \in L^2_{0,q}(\Omega_c, E)$, there exists a form $v \in L^2_{0,q-1}(\Omega, E)$ such that $f = \overline{\partial}_{q-1} v$ on Ω. By Theorem 3.4.6 in [H65], there exists $\tilde{v}_1 \in L^2_{0,q-1}(\Omega_{c_2}, E)$ such that $f = \overline{\partial}_{q-1}\tilde{v}_1$ on Ω_{c_2}. Since $\overline{\partial}_{q-1}(u - \tilde{v}_1) = 0$ on Ω_{c_1}, by Theorem 3.4.7 in [H65], there exists $\hat{v}_1 \in L^2_{0,q-1}(\Omega_{c_2}, E)$ such that $\overline{\partial}_{q-1}\hat{v}_1 = 0$ on Ω_{c_2} and $\|u - \tilde{v}_1 - \hat{v}_1\|^2_{\Omega_{c_1}} < 1/2$. Let $v_1 = \tilde{v}_1 + \hat{v}_1$. Continuing this procedure inductively, we then obtain $v_j \in L^2_{0,q-1}(\Omega_{c_{j+1}}, E)$ such that $f = \overline{\partial}_{q-1} v_j$ on $\Omega_{c_{j+1}}$ and

$$\|v_j - v_{j+1}\|_{\Omega_{c_{j+1}}} \leq \frac{1}{2^j}.$$

It then follows that there exists $v \in L^2_{0,q}(\Omega, E)$ such that for any k, $\|v_j - v\|_{\Omega_{c_k}} \to 0$. Hence $f = \overline{\partial}_{q-1} v$ on Ω.

Remark. Recall that a domain $\Omega \subset\subset X$ is said to satisfies property (P) in the sense of Catlin [Ca84b] if for any $M > 0$, there exists a neighborhood U of $b\Omega$ and a function $f \in C^\infty(\Omega \cap U)$ such that $|f| \leq 1$ and

$$L_f(z,\xi) \geq M|\xi|^2_h$$

[2]Although the results in [H65] is stated for only forms with values in the trivial line bundle, it is obvious that they also hold for forms with values in any holomorphic line bundle.

for all $z \in \Omega \cap U$ and $\xi \in T_z^{1,0}(X)$.

It is well-known that any relatively compact smoothly bounded pseudoconvex domain with finite type boundary in a complex surface satisfies property (P)(see Catlin [Ca89] and Fornaess-Sibony [FoS89][3]). Hence Theorem 1.1 applies to these domains.

4. Pseudoconcavity and the asymptotic estimates

Let $\Omega \subset\subset X$ be a smoothly bounded domain in a complex manifold X of dimension n.

Definition 4.1. A point $q \in b\Omega$ is an Andreotti pseudoconcave point if there exists a fundamental system of neighborhoods $\{U\}$ such that q is an interior point of each of the sets

$$(\widehat{U \cap \Omega})_U = \{p \in U | \ |f(p)| \leq \sup_{z \in U \cap \Omega} |f(z)|, \forall f \in \mathcal{O}(U)\}.$$

A domain Ω is Andreotti pseudoconcave if each of its boundary points is.

Note that this definition does not depend on the choice of the fundamental system of neighborhoods.

We list some well known properties of Andreotti pseudoconcavity.

(1) There are no relatively compact pseudoconcave domains in $\mathbb{C}^n$.
(2) If the Levi form of $b\Omega$ has at each $q \in b\Omega$ at least one negative eigenvalue, then Ω is (strictly) pseudoconcave.
(3) If X has a relatively compact pseudoconcave subdomain, then $\mathcal{O}(X) = \mathbb{C}$.
(4) Each complex submanifold of $\mathbb{CP}^n$ has a pseudoconcave neighborhood.

Recall that a real hypersurface M in X is minimal (in the sense of Trepreau) at a point q if there does not exist the germ of complex hypersurface passing through q and contained in M. We say an open set is minimal at a boundary point if the boundary is minimal at that point.

Theorem 4.1. *Let $q \in b\Omega$. If the Levi form has no positive eigenvalues in a neighborhood of q and Ω is minimal at q, then q is an Andreotti pseudoconcave point.*

Theorem 4.2. *If Ω is Andreotti pseudoconcave and relatively compact and if E is a holomorphic line bundle over $\overline{\Omega}$, then there is some constant depending only on Ω and E such that*

$$h^0(\Omega, E^k) \leq Ck^n,$$

where as before $h^0(\Omega, E^k)$ is the dimension of the space of global holomorphic sections over Ω with values in E^k.

The first theorem will follow directly from some well-known results. When $b\Omega$ is real analytic, this theorem is essentially contained in [BFe78]. The second theorem is due to Siegel [Sie55] and Andreotti [And63] and [And73]. We sketch the proof from [And63], simplified to apply to manifolds rather than spaces.

We state Trepreau's Theorem [Tre86]

[3]Both papers stated their results for domains in $\mathbb{C}^2$. The construction of Fornaess and Sibony can be easily seen to work on complex manifolds as well. Catlin [Ca87] also constructed plurisubharmonic function with large complex hessian near the boundary for a smooth bounded pseudoconvex domain of finite type in $\mathbb{C}^n$. As a consequence, such domains satisfy property (P). His construction should also work for domains in complex manifolds as well.

Theorem 4.3. *If $b\Omega$ is minimal at the point q then there exist a fundamental system of neighborhoods $\{V\}$ of q and an open set S lying on one side of $b\Omega$, with $bS \cap b\Omega$ an open neighborhood of q in $b\Omega$, such that*

$$\mathcal{O}(V) \to \mathcal{O}(V \cap S)$$

is surjective.

To prove Theorem 4.1, we first show that the set S from Trepreau's theorem lies in Ω. To see this, note that as long as ϵ is small enough, $b\Omega \cap B(q,\epsilon)$ is Levi pseudoconvex, as part of the boundary of $\Omega^c \cap B(q,\epsilon)$ and so $\Omega^c \cap B(q,\epsilon)$ is pseudoconvex and therefore a domain of holomorphy. Thus for no neighborhood V of q is the map

$$\mathcal{O}(V) \to \mathcal{O}(V \cap \Omega^c)$$

surjective. So $S \subset \Omega$ and Trepreau's theorem asserts that each holomorphic function on $V \cap \Omega$, for V in the fundamental family of neighborhoods , extends holomorphically to V. It then follows that

$$\sup_V |f| = \sup_{V\cap\Omega} |f|$$

and so q is a pseudoconcave point. We thus conclude the proof of Theorem 4.1.

Let $\Omega \subset\subset X$. We assume that Ω is Andreotti pseudoconcave . We may extend the relevant transition functions of the bundle E holomorphically across the boundary of Ω and so there is no loss of generality in assuming that E is a holomorphic line bundle over an open neighborhood of $\overline{\Omega}$.

The following lemma (see [Sie55]) is fundamental to the arguments.

Lemma 4.4. *There exist a finite number of points $x_a \in \Omega$, $a = 1 \dots N$ and an integer h such that the only section of E which vanishes to at least order h at each x_a is the zero section. Further, there exists a constant C depending only on a fixed covering of Ω so that h can be chosen to be any integer greater than $C \ln g$, where g depends only on bounds for the transition functions for E.*

Proof. Let

$$P_r = \{z \in \mathbb{C}^n \mid \ |z_m| < r, \ \ 1 \le m \le n\}$$

be a polydisc. There exist finite open coverings, $\{\Omega_k, \ k = 1, \dots, K\}$ and $\{W_a, \ a = 1, \dots, N\}$, of $\overline{\Omega}$ with the following properties.

(1) Each Ω_k is diffeomorphic to the unit ball in $\mathbb{C}^n$ and biholomorphic to a domain of holomorphy in $\mathbb{C}^n$. Note that any holomorphic line bundle over Ω_k is holomorphically trivial.

(2) There is some real number r_0, $0 < r_0 < 1$ such that for each a there exists a biholomorphism ϕ_a defined on an open neighborhood of $\overline{W_a}$ taking $W_a \to P_{r_0}$. It follows that there is some real number r_1, not depending on a and with $r_0 < r_1 < 1$, for which $\phi_a^{-1}(P_{r_1})$ is defined. Set

$$V_a = \phi_a^{-1}(P_{r_1}).$$

(3) There exists a map

$$J : \{1, \dots, N\} \to \{1, \dots, K\}$$

such that

$$V_a \subset \Omega_{J(a)} \cap \{\widehat{\Omega_{J(a)} \cap \Omega}\}.$$

Choose a nowhere zero holomorphic section $\sigma_j : \Omega_j \to E|_{\Omega_j}$. Define

$$g_{jk} : \Omega_j \cap \Omega_k \to C$$

by

$$\sigma_j = g_{jk}(x)\sigma_k.$$

Let

$$g = \sup_{j,k} \sup_{x \in \Omega_j \cap \Omega_k} |g_{jk}(x)|.$$

Note that $1 \leq g < \infty$.

Set $\Omega_0 = \bigcup V_a$. For each section $s : \Omega_0 \to E$ we introduce the notation

$$s|_{V_a} = s_{J(a)}\sigma_{J(a)}$$

and define

$$\begin{aligned} ||s||_V &= \sup\nolimits_{a,V_a} |s_{J(a)}| \\ ||s||_W &= \sup\nolimits_{a,W_a} |s_{J(a)}| \end{aligned}$$

Note that

$$s_{J(b)} = s_{J(a)} g_{J(a)J(b)}.$$

Since

$$V_{J(a)} \subset \widehat{(\Omega_{J(a)} \cap \Omega)}$$

we have

$$\sup_{V_{J(a)}} |s_{J(a)}| \leq \sup_{\Omega_{J(a)} \cap \Omega} |s_{J(a)}|. \tag{4.1}$$

Let $x \in \Omega_{J(a)} \cap \Omega$. Then, because $\{W_b\}$ is a covering of Ω, there is some b for which $x \in W_b$. So

$$|s_{J(a)}(x)| = |g_{J(a)J(b)}(x) s_{J(b)}(x)| \leq g\ |s_{J(b)}(x)|.$$

Combining this with (4.1), we obtain

$$||s||_V \leq g\ ||s||_W.$$

Note how the pseudoconcavity was used to derive this inequality. Now we need a good bound for $||s||_W$ in terms of $||s||_V$. This is the main point in the proof. Let

$$x_a = \phi_a^{-1}(0).$$

We now make use of the hypothesis that s vanishes to order h at each x_a. Recall the following version of the Schwarz Lemma [Sie55].

Lemma 4.5. *If $F(z)$ is holomorphic on P_{r_1} and vanishes to order at least h at the origin, then*

$$\sup_{z \in P_{r_0}} |F(z)| \leq \left(\frac{r_0}{r_1}\right)^h \sup_{z \in P_{r_1}} |F(z)|.$$

Let $q = r_0/r_1$. Applying the Schwarz Lemma to $W_a \subset V_a$ for each a, we obtain

$$||s||_W \leq q^h ||s||_V.$$

Note that q satisfies $0 < q < 1$ and that q only depends on the choice of $\{W_a\}$ and $\{\phi_a\}$.

We now have

$$||s||_V \leq g||s||_W \leq q^h g||s||_V$$

and so $s \equiv 0$ provided we take h to be an integer satisfying

$$h > -\frac{\ln g}{\ln q}. \tag{4.2}$$

(Recall $g \geq 1$ and $0 < q < 1$.) $\square$

It is now easy to see how the Fundamental Lemma implies Theorem 4.2. Let $\Gamma(\overline{\Omega}, E)$ be the complex vector space of holomorphic sections of E over some neighborhood of Ω. The neighborhood is allowed to depend on the section. Let $J(x_a)$ be the space of jets up to order h at x_a of holomorphic sections of E. Consider the map

$$\Gamma(\overline{\Omega}, E) \to \oplus_{a=1}^{N} J(x_a).$$

The Fundamental Lemma tells us that this map is injective. The dimension of each $J(x_a)$ is $\left(\begin{array}{c} n+h \\ h \end{array}\right)$.

Thus

$$h^0(\Omega, E) \leq N \left(\begin{array}{c} n+h \\ h \end{array}\right).$$

We want to estimate the right hand side when h is large. We do this using Sterling's asymptotic formula:

$$m! \approx \sqrt{2\pi m}\frac{m^m}{e^m}.$$

So, as $h \to \infty$

$$\begin{aligned} \left(\begin{array}{c} n+h \\ h \end{array}\right) = \frac{(n+h)!}{h!n!} &\approx \frac{\sqrt{1+\frac{n}{h}}(n+h)^{n+h}}{e^n h^h n!} \\ &\approx \frac{(n+h)^n}{n!} \\ &\approx \frac{h^n}{n!}. \end{aligned}$$

Now we replace E by E^k. This means that the transition functions are replaced by their k^{th} powers and so g becomes g^k. Then h is replaced by $c'k$ for some c' depending on E and so

$$h^0(\Omega, E^k) < C \left(\begin{array}{c} n+c'k \\ c'k \end{array}\right) < C'k^n. \tag{4.3}$$

This completes the proof, based on [And63], of Theorem 4.2.

We will need some minor modifications of Lemma 4.4. First we replace E^k by a bundle of the form $L^k \otimes F^s$. The transition functions for $L^k \otimes F^s$ are of the form $g_{ij}^k f_{ij}^s$ and so are bounded by C^{k+s} for some C. So in the proof of Theorem 4.2 the inequality (4.2) is replaced by

$$h > \frac{-\ln C^{k+s}}{\ln q} \tag{4.4}$$

and inequality (4.3) is replaced by

$$h^0(\Omega, L^k \otimes F^s) < C \left(\begin{array}{c} n+c'(k+s) \\ c'(k+s) \end{array}\right) < C'(k+s)^n. \tag{4.5}$$

Next we assume that local holomorphic coordinates $\zeta_1, \ldots, \zeta_n$ are specified in a neighborhood of each x_a and that the only sections we consider are those that in a neighborhood of x_a are holomorphic functions of only $\zeta_1, \ldots, \zeta_m$ for some $m \leq n$. Denote the space of such sections by $\Gamma_0(\overline{\Omega}, E)$. We have restricted the set of sections so of course it still follows that a section vanishing to at least order h at each x_a must be identically zero. At each x_a there are $\begin{pmatrix} m+h \\ h \end{pmatrix}$ polynomials in m variables of degree less than or equal to h. So

$$dim\ \Gamma_0(\overline{\Omega}, E) \leq N \begin{pmatrix} m+h \\ h \end{pmatrix}$$

and as before

$$dim\ \Gamma_0(\overline{\Omega}, E^k) \leq C \begin{pmatrix} m+c'k \\ c'k \end{pmatrix} \leq C'k^m.$$

And finally, we combine these two modifications.

$$dim\ \Gamma_0(\overline{\Omega}, L^k \otimes F^s) \leq C'(k+s)^m. \tag{4.6}$$

We conclude with an application from [Sie55]. Let X be a compact complex manifold with $dim\ X = n$. (Or, more generally, let X contain a relatively compact Andreotti pseudoconcave subset, see [And63]. In particular, the results will apply to any X containing a set Ω as in Theorem 4.1) . Recall that meromorphic functions on a complex manifold X are analytically dependent if

$$df_1 \wedge \ldots \wedge df_m = 0$$

at each point of X at which the functions are all holomorphic. And they are algebraically dependent if there is a nontrivial polynomial P over C with

$$P(f_1, \ldots, f_m) = 0$$

at all such points. It is easy to see that algebraic dependence implies analytic dependence. Here is the converse.

Theorem 4.6. *If the meromorphic functions $f_1, \ldots, f_m$ on X are analytically dependent, then they are also algebraically dependent.*

This implies that the field of meromorphic functions on X is an algebraic extension of the field of rational functions in d variables, with $d \leq n$. Thus

$$K(X) \cong Q(t_1, \ldots, t_d, \theta),\ \theta \text{ algebraic in } t_1, \ldots, t_d.$$

We first relate memomorphic functions to line bundles. Given a meromorphic function f, we may find a finite covering

$$X = \bigcup U_j$$

and holomorphic functions over U_j such that

$$f = \frac{p_j}{q_j} \text{ on } U_j, \text{ and } \frac{p_j}{q_j} = \frac{p_k}{q_k} \text{ on } U_j \cap U_k.$$

Let L be the line bundle with transition functions

$$g_{jk} = \frac{q_j}{q_k} \in \mathcal{O}^*(U_j \cap U_k).$$

Then

$$p = \{p_j\} \text{ and } q = \{q_j\}$$

are global sections of L and $f = p/q$ is a global quotient.

We are now ready to prove Theorem 4.6. We change notation and start with analytically independent meromorphic functions $f_1, \ldots, f_m$ and a meromorphic function f with

$$df_1 \wedge \ldots \wedge df_m = 0$$

at each point where this makes sense. We need to find a polynomial such that $P(f_1, \ldots f_m, f) = 0$ at each point where the functions are all holomorphic. Let L_j be the bundle associated to f_j and let F be the bundle associated to f. Then each f_j is a global quotient of sections of

$$L = L_1 \otimes L_2 \otimes \ldots \otimes L_m.$$

Further, $f_1^{k_1} \cdots f_m^{k_m} f^s$ is a global quotient of $L^k \otimes F^s$ where $k = k_1 + \ldots + k_m$.

We write

$$\begin{aligned} f_j &= \frac{s_j}{s_0} & & s_j, s_0 \text{ global sections of } L, \\ f &= \frac{\phi}{\psi} & & \phi, \psi \text{ global sections of } F. \end{aligned}$$

We fix some positive integers r and s. Let

$$\begin{aligned} W_0(r,s) = \{&\text{polynomials of degree at most } r \\ &\text{in each of } X_1, \ldots, X_m \\ &\text{and degree at most } s \text{ in } X_{m+1}\} \end{aligned}$$

We want to eliminate the denominators in our global quotients and also to work only with homogeneous polynomials. So let

$$W(r,s) = \{Q(\xi, \eta, X_1, \ldots, X_m, X_{m+1}) = \xi^{mr}\eta^s P(\frac{X_1}{\xi}, \ldots, \frac{X_m}{\xi}, \frac{X_{m+1}}{\eta}), P \in W_0(r,s)\}.$$

Thus $Q \in W(r,s)$ is homogeneous in the sense that

$$Q(a\xi, b\eta, aX_1, \ldots, aX_m, bX_{m+1}) = a^{mr} b^s Q(\xi, \eta, X_1, \ldots, X_m, X_{m+1}).$$

We may assume that at the points x_a in the proof of Theorem 4.2

$$df_1 \wedge \ldots \wedge df_m \neq 0.$$

So these functions define a partial set of local coordinates which we use to define $\Gamma_0(\overline{\Omega}, L^{mr} \otimes F^s)$.

Next define

$$\Pi : W(r,s) \to \Gamma_0(X, L^{mr} \otimes F^s)$$

by

$$\Pi Q = Q(s_0, \psi, s_1, \ldots, s_m, \phi).$$

It suffices to prove that Π is not injective. The modifications of Theorem 4.2 apply as long as (4.4) holds. Thus (4.6) holds with k replaced by mr:

$$dim\ \Gamma_0(X, L^{mr} \otimes F^s) \leq C'(mr + s)^m.$$

It is easy to see that

$$dim\ W(r,s) = (r+1)^m (s+1).$$

So if r and s can be chosen such that

$$(r+1)^m (s+1) > C'(mr+s)^m$$

then Π is not injective. We write this inequality as

$$s+1 > C\frac{(m+\frac{s}{r})^m}{(1+\frac{1}{r})^m}. \tag{4.7}$$

We first choose s so that

$$s+1 > 2Cm^m$$

and then choose r large enough to guarantee (4.7).

5. Hearing the finite type condition in two dimensions

5.1. **The finite type condition.** Hereafter, we will assume that X is a complex surface and Ω is a relatively compact domain with smooth boundary in X. The boundary $b\Omega$ is said to be of finite type (in the sense of D'Angelo [D82]) if the normalized order of contact of any analytic variety with $b\Omega$ is finite. The highest order of contact is the type of the domain.

Assume that X is equipped with a hermitian metric h. Let $r(z)$ be the signed geodesic distance from z to $b\Omega$ such that $r<0$ on Ω and $r>0$ outside of Ω. Then r is smooth on a neighborhood U of Ω and $|dr|_h = 1$ on U. Let $z' \in b\Omega$ and let L be a normalized $(1,0)$-vector field in a neighborhood of z' such that $Lr=0$. For any integers $j,k\geq 1$, let

$$\mathcal{L}_{jk}\partial\overline{\partial}r(z') = \underbrace{L\dots L}_{j-1 \text{ times}}\ \underbrace{\overline{L}\dots\overline{L}}_{k-1\text{ times}}\ \partial\overline{\partial}r(L,\overline{L})(z'),$$

Let m be any positive integer. For any $2\leq l\leq 2m$, let

$$A_l(z') = \Big(\sum_{\substack{j+k\leq l\\ j,k>0}} |\mathcal{L}_{jk}\partial\overline{\partial}r(z')|^2\Big)^{1/2}. \tag{5.1}$$

For any $\tau>0$, let

$$\delta(z',\tau) = \sum_{l=2}^{2m} A_l(z')\tau^l. \tag{5.2}$$

It is easy to see that

$$\delta(z',\tau)\lesssim \tau^2 \quad\text{and}\quad c^{2m}\delta(z',\tau)\leq \delta(z',c\tau)\leq c^2\delta(z',\tau), \tag{5.3}$$

for any τ and c such that $0<\tau, c<1$. Furthermore, $b\Omega$ is of finite type $2m$ if and only if $\delta(z',\tau)\gtrsim \tau^{2m}$ uniformly for all $z'\in b\Omega$ and $\delta(z_0',\tau)\lesssim \tau^{2m}$ for some $z_0'\in b\Omega$. (Here and throughout the paper, $f\lesssim g$ means that $f\leq Cg$ for some positive constant C. It should be clear from the context which parameters the constant C is independent of. For example, the constant in (5.3) is understood to be independent of z' and τ.)

Let z^0 be a fixed boundary point and let V be a neighborhood of z^0 such that its closure is contained in a coordinate patch. Let m be any positive integer. It follows from Proposition 1.1 in [FoS89] that for any $z'\in V\cap b\Omega$, after a possible shrinking of V, there exists a neighborhood $U_{z'}$ of z' and local holomorphic coordinates (z_1,z_2) centered at z' and depending smoothly on z' such that in these coordinates

$$U_{z'}\cap\Omega = \{z\in U_{z'} \mid \rho(z) = \text{Re } z_2 + \psi(z_1, \text{Im } z_2) < 0\},$$

where $\psi(z_1,\text{Im } z_2)$ has the form of

$$\psi(z_1,\text{Im } z_2) = P(z_1) + (\text{Im } z_2)Q(z_1) + O\big(|z_1|^{2m+1} + |\text{Im } z_2||z_1|^{m+1} + |\text{Im } z_2|^2|z_1|\big) \tag{5.4}$$

with

$$P(z_1)=\sum_{l=2}^{2m}\sum_{\substack{j+k=l\\ j,k>0}} a_{jk}(z')z_1^j\bar z_1^k \quad \text{and} \quad Q(z_1)=\sum_{l=2}^{m}\sum_{\substack{j+k=l\\ j,k>0}} b_{jk}(z')z_1^j\bar z_1^k$$

being polynomials without harmonic terms. Furthermore, there exist positive constants C_1 and C_2, independent of z', such that

$$C_1A_l(z')\le\Big(\sum_{\substack{j+k\le l\\ j,k>0}}|a_{jk}(z')|^2\Big)^{1/2}\le C_2A_l(z')$$

for $2\le l\le 2m$.

The above properties hold without the pseudoconvex or finite type assumption on Ω. Under the assumption that $b\Omega$ is pseudoconvex of finite type $2m$, it then follows from Proposition 1.6 in [FoS89] that for all $0<\tau<1$,

$$\sum_{l=2}^{m}B_l(z')\tau^l\lesssim\tau(\delta(z',\tau))^{1/2}, \tag{5.5}$$

where

$$B_l(z')=\Big(\sum_{\substack{j+k\le l\\ j,k>0}}|b_{jk}(z')|^2\Big)^{1/2}.$$

The anisotropic bidisc $R_\tau(z')$ is given in the (z_1,z_2)-coordinates by

$$R_\tau(z')=\{|z_1|<\tau,|z_2|<\delta(z',\tau)^{1/2}\}. \tag{5.6}$$

We refer the reader to [Ca89, Mc89, NRSW89, Fu05b] and references therein for a discussion of these and other anisotropic "balls". It was shown in [Fu05b] (see Lemmas 3.2 and 3.3 therein) that the anisotropic bidiscs $R_\tau(z')$ satisfy the following doubling and engulfing properties: There exists a positive constant C, independent of z', such that if $z''\in R_\tau(z')\cap b\Omega$, then $C^{-1}\delta(z',\tau)\le\delta(z'',\tau)\le C\delta(z',\tau)$, $R_\tau(z')\subset R_{C\tau}(z'')$, and $R_\tau(z'')\subset R_{C\tau}(z')$.

5.2. **Interior estimates.** Let E be a holomorphic line bundle over Ω that extends smoothly to the boundary $b\Omega$. Let $e_k(\lambda;z,w)$ be the spectral kernel of the $\bar\partial$-Neumann Laplacian on $(0,1)$-forms on Ω with values in E^k. Let $\pi\colon U\to b\Omega$ be the projection onto the boundary such that $|r(z)|=\text{distance}\,(z,\pi(z))$. Shrinking U if necessary, we have $\pi\in C^\infty(U)$. Write $\tau_k=1/\sqrt{k}$.

Proposition 5.1. *For any $C,c>0$,*

$$\text{tr }e_k(Ck;z,z)\lesssim k(\delta(\pi(z),\tau_k))^{-1}, \tag{5.7}$$

for all sufficiently large k and all $z\in\Omega$ with $d(z)\ge c(\delta(\pi(z),\tau_k))^{1/2}$.

It is a consequence of a classical result of Gärding [G53] that for any compact subset K of Ω,

$$\text{tr }e_k(Ck;z,z)\lesssim k^2,\qquad \text{for } z\in K. \tag{5.8}$$

Evidently, the constant in the above estimate depends on K. (See Theorem 3.2 in [Ber04] for a more general and precise version of this result.) In fact, this is also true for any $z\in\Omega$ with $d(z)\ge 1/\sqrt{k}$. (Compare estimate (2.3) in [Me81], Theorem 3.2 in [Ber04], and Proposition 5.7 in [Ber05].) Therefore, it suffices to establish (5.7) on $\{z\in\Omega, c(\delta(\pi(z),\tau_k))^{1/2}\le$

$|r(z)| \le \tau_k\}$. This also follows from the elliptic theory, via an anisotropic rescaling. We provide details below.

Let $z' \in b\Omega$. Following the discussion in Section 5.1, we can choose holomorphic coordinates centered and orthonormal at z' such that in a neighborhood $U_{z'}$ of z', $b\Omega$ is defined by $\rho(z_1, z_2) = \mathrm{Re}\; z_2 + \psi(z_1, \mathrm{Im}\; z_2)$ where $\psi(z_1, \mathrm{Im}\; z_2)$ is in the form of (5.4). Assume that the hermitian metric is given on $U_{z'}$ by

$$h = \frac{i}{2}\sum_{j,l=1}^{2} h_{jl}(z)dz_j \wedge d\bar{z}_l, \quad \text{with} \quad h_{jl}(0) = \delta_{jl} \tag{5.9}$$

and the fiber metric on E is given by

$$|e(z)|^2 = e^{-\varphi(z)} \qquad \text{with} \quad \varphi(z) = \sum_{j,l=1}^{2} a_{jl} z_j \bar{z}_l + O(|z|^3), \tag{5.10}$$

where $e(z)$ is an appropriate holomorphic frame of E over $U_{z'}$.

Write $\Omega_{z'} = \Omega \cap U_{z'}$. Let $\omega_2' = \partial\rho$ and $\omega_1' = \rho_{\bar{z}_2} dz_1 - \rho_{\bar{z}_1} dz_2$. Let ω_2 and ω_1 be the orthonormal basis for $(1,0)$-forms on $\Omega_{z'}$ obtained by applying the Gram-Schmidt process to ω_2' and ω_1'. Let L_2 and L_1 be the dual basis for $T^{1,0}(\Omega_{z'})$.

Write $\delta_k = \delta(z', \tau_k)$ and let $c > 0$. For any σ_k such that $\tau_k \ge \sigma_k \ge c\delta_k^{1/2}$, we define the anisotropic dilation $(z_1, z_2) = F_k(\zeta_1, \zeta_2) = (\tau_k \zeta_1,\ \sigma_k \zeta_2)$. Let $U_{z'}^k = F_k^{-1}(U_{z'})$ and $\Omega_{z'}^k = F_k^{-1}(\Omega_{z'})$. On $\Omega_{z'}^k$, we use the base metric given by

$$h^{(k)} = \frac{i}{2}\sum_{j,l=1}^{2} h_{jl}(\tau_k\zeta_1, \sigma_k\zeta_2)d\zeta_j \wedge d\bar{\zeta}_l$$

and on $E^{(k)} = F_{k*}(E^k)$ we use the fiber metric given by the weight function

$$\varphi^{(k)}(\zeta) = k\varphi(\tau_k\zeta_1, \sigma_k\zeta_2).$$

Note that $\Omega_{z'}^k = \{(\zeta_1, \zeta_2) \in U_{z'}^k \mid \rho_k(\zeta_1, \zeta_2) < 0\}$, where $\rho_k(\zeta_1, \zeta_2) = (1/\sigma_k)\rho(\tau_k z_1, \sigma_k z_2)$. Let ω_1^k and ω_2^k be the orthonormal basis for $(1,0)$-forms on $\Omega_{z'}^k$ obtained as in the proceeding paragraph but with ρ replaced by ρ_k and (z_1, z_2) replaced by (ζ_1, ζ_2) respectively. Let L_1^k and L_2^k be the dual basis for $T^{1,0}(\Omega_{z'}^k)$. We define $\mathcal{F}_k \colon L^2(\Omega_{z'}, E^k) \to L^2(\Omega_{z'}^k, E^{(k)})$ by

$$\mathcal{F}_k(v)(\zeta_1, \zeta_2) = (\tau_k\sigma_k)v(\tau_k\zeta_1, \sigma_k\zeta_2)$$

and extend $\mathcal{F}_k$ to act on forms by acting componentwise as follows:

$$\mathcal{F}_k(v_1\overline{\omega}_1 + v_2\overline{\omega}_2) = \mathcal{F}_k(v_1)\overline{\omega}_1^k + \mathcal{F}_k(v_2)\overline{\omega}_2^k, \quad \mathcal{F}_k(v\overline{\omega}_1 \wedge \overline{\omega_2}) = \mathcal{F}_k(v)\overline{\omega}_1^k \wedge \overline{\omega}_2^k.$$

(Hereafter, we identify a form with values in E^k with its representation in the given local holomorphic trivialization.) It is easy to see that $\mathcal{F}_k$ is isometric on L^2-spaces with respect to specified metrics:

$$\|u\|_{h,k\varphi}^2 = \|\mathcal{F}_k u\|_{h^{(k)},\varphi^{(k)}}^2,$$

where $\|\cdot\|_{h,k\varphi}$ denotes the L^2-norm with respect to the base metric h and the fiber metric $k\varphi$ and likewise $\|\cdot\|_{h^{(k)},\varphi^{(k)}}$ the L^2-norm with respect to the base metric $h^{(k)}$ and the fiber metric $\varphi^{(k)}$. Let

$$Q^{(k)}(u, v) = \tau_k^2 Q_\Omega^k(\mathcal{F}_k^{-1}u, \mathcal{F}_k^{-1}v)$$

with $\mathcal{D}(Q^{(k)}) = \{v \mid \mathcal{F}_k^{-1}v \in \mathcal{D}(Q^k_\Omega)$ and Supp $\mathcal{F}_k^{-1}v \subset U_{z'}\}$, where $Q^k_\Omega = Q^{E^k}_{\Omega,1}$ is the sesquilinear form associated with the $\overline{\partial}$-Neumann Laplacian $\Box^k_\Omega = \Box^{E^k}_{\Omega,1}$ on $(0,1)$-forms on Ω with values in E^k. Let $\Box^{(k)}$ be the self-adjoint operator associated with $Q^{(k)}$.

Let $P' = \{\zeta \in U^k_{z'} \mid |\zeta_1| < 1/2, |\zeta_2 + 1| < 1/2\}$. It is easy to see that for sufficiently large $k > 0$, P' is a relatively compact subset of $\Omega^k_{z'}$.

Lemma 5.2. *Let $u \in \mathcal{D}(Q^{(k)}) \cap C^\infty_c(P')$. Then*

$$\|u\|^2_{1,\Omega^k_{z'}} \lesssim Q^{(k)}(u,u) + \|u\|^2_{h^{(k)},\varphi^{(k)}}, \tag{5.11}$$

where $\|\cdot\|_{1,\Omega^k_{z'}}$ is the L^2-Sobolev norm of order 1 on $\Omega^k_{z'}$.

Proof. Write $u = u_1\overline{\omega}_1^{(k)} + u_2\overline{\omega}_2^{(k)}$ and $v = \mathcal{F}_k^{-1}(u) = v_1\overline{\omega}_1 + v_2\overline{\omega}_2$. Since v is supported on $R'_k = F_k(P') = \{z \in U_{z'} \mid |z_1| < (1/2)\tau_k, |z_2 + \sigma_k| < (1/2)\sigma_k\} \subset\subset \Omega$, it follows from integration by parts that

$$Q^k_\Omega(v,v) + k\|v\|^2_{h,k\varphi} \gtrsim \sum_{j,l=1}^2 \|\overline{L}_j v_l\|^2_{h,k\varphi}.$$

Notice that on R'_k,

$$|h_{jl}(z) - \delta_{jl}| \lesssim \tau_k.$$

¿From Section 5.1, we obtain by direct calculations that on R'_k,

$$\overline{L}_1 = (1 + O(\tau_k))\frac{\partial}{\partial\bar{z}_1} + O(\tau_k)\frac{\partial}{\partial\bar{z}_2}, \quad \overline{L}_2 = O(\tau_k)\frac{\partial}{\partial\bar{z}_1} + (1 + O(\tau_k))\frac{\partial}{\partial\bar{z}_2}.$$

Thus,

$$\begin{aligned} Q^{(k)}(u,u) + \|u\|^2_{h^{(k)},\varphi^{(k)}} &= \tau_k^2(Q^k_\Omega(v,v) + k\|v\|^2_{h,k\varphi}) \gtrsim \sum_{j,l=1}^2 \|\tau_k\overline{L}_j v_l\|^2_{h,k\varphi} \\ &\gtrsim \sum_{j=1}^2 \big(\|\frac{\partial u_j}{\partial\bar{z}_1}\|^2_{h^{(k)},\varphi^{(k)}} + \frac{\tau_k^2}{\sigma_k^2}\|\frac{\partial u_j}{\partial\bar{z}_2}\|^2_{h^{(k)},\varphi^{(k)}}\big) \geq \sum_{j,l=1}^2 \|\frac{\partial u_j}{\partial\bar{z}_l}\|^2_{h^{(k)},\varphi^{(k)}}. \end{aligned}$$

Since $|\varphi^{(k)}| \lesssim 1$ on P' and u is compactly supported in P', a simple integration by parts argument then yields the estimate (5.11). ☐

We now complete the proof of Proposition 5.1. From Lemma 5.2, we know that $\Box^{(k)}$ is uniformly (independent of k) strong elliptic on P'. Let $P'' = \{\zeta \in P' \mid |\zeta_1| < 1/4, |\zeta_2+1| < 1/4\}$. Thus by Gärding's inequality,

$$\|u\|_{2M,P''} \lesssim \|(\Box^{(k)})^M u\|_{P'} + \|u\|_{P'} \tag{5.12}$$

for any $u \in C^\infty_{0,1}(\Omega^k_{z'}, E^{(k)})$, where $\Box^{(k)} = \tau_k^2\mathcal{F}_k\Box^k_\Omega\mathcal{F}_k^{-1}$ acts formally.

Let $\mathbf{E}_k(\lambda)$ be the spectral resolution of $\Box^k_\Omega$, the $\overline{\partial}$-Neumann Laplacian on Ω on $(0,1)$-forms with values in E^k. Let $v \in \mathbf{E}_k(Ck)(L^2_{0,1}(\Omega, E^k))$ be of unit norm. Then for any positive integer M,

$$\|(\Box^k_\Omega)^M v\|^2_{\Omega,E^k} \leq (Ck)^{2M}. \tag{5.13}$$

Let $u_k = \mathcal{F}_k(v')$, where v' is the restriction of v to $U_{z'}$. Then

$$(\Box^{(k)})^M u_k = \tau_k^{2M}\mathcal{F}_k(\Box^k_\Omega)^M v'.$$

By (5.13), we have

$$\|(\Box^{(k)})^M u_k\|^2_{h^{(k)},\varphi^{(k)}} \lesssim 1.$$

We obtain from (5.12) and the Sobolev embedding theorem that

$$\tau_k\sigma_k|v(0,-\sigma_k)| = |u_k(0,-1)| \lesssim 1.$$

Thus by (2.1), we have

$$\text{tr } e_k(Ck;(0,-\sigma_k),(0,-\sigma_k)) \lesssim (\tau_k\sigma_k)^{-2} \leq k\delta_k^{-1}.$$

Since the constant in this estimate is uniform as z' varies the boundary $b\Omega$ and σ_k varies between $c\delta_k^{1/2}$ and τ_k, we thus conclude the proof of Proposition 5.1.

5.3. **Boundary estimates.**

5.3.1. *Main boundary estimate.* We shall establish the following boundary estimate for the spectral kernel.

Proposition 5.3. *Let $C > 0$. For any $z' \in b\Omega$ and sufficiently large k,*

$$\int_{R_{\tau_k}(z')\cap\Omega} \text{tr } e_k(Ck;z,z)dV(z) \lesssim (\delta(z',\tau_k))^{-1/2}. \tag{5.14}$$

Recall that $\tau_k = 1/\sqrt{k}$ and $R_{\tau_k}(z')$ is the anisotropic bidisc given by (5.6). Assume Proposition 5.3 for a moment, we now prove the sufficiency in Theorem 1.3. In fact we shall prove the following:

Proposition 5.4. *Let $\Omega \subset\subset X$ be a smoothly bounded pseudoconvex domain in a complex surface. Let E be a holomorphic line bundle over Ω that extends smoothly to $b\Omega$. If $b\Omega$ is of finite type $2m$, then for any $C > 0$, there exists $C' > 0$ such that $N_k(Ck) \leq C'k^{1+m}$. More precisely, $\lim_{k\to\infty} N_k(Ck)/k^{1+m} = 0$ when $m > 1$.*

Proof. We cover $b\Omega$ by finitely many open sets, each of which is contained in a coordinate patch as the V's in Section 5.1. Let $z' \in V \cap b\Omega$. Multiplying both sides of (5.14) by $(\delta(z',\tau_k))^{-1/2}$ and integrating with respect to $z' \in V \cap b\Omega$, we obtain by the Fubini-Tonelli theorem that

$$\int_\Omega \text{tr } e_k(Ck;z,z)\,dV(z)\int_{V\cap b\Omega}\chi_{\Omega\cap R_{\tau_k}(z')}(z)(\delta(z',\tau_k))^{-1/2}\,dS(z') \lesssim \int_{V\cap b\Omega}(\delta(z',\tau_k))^{-1}\,dS(z').$$

(Here χ_S denotes the characteristic function of the set S.) By Lemma 3.4 in [Fu05], we then have

$$\int_{V\cap\{z\in\Omega|d(z)<c(\delta(\pi(z),\tau_k))^{1/2}\}} \text{tr } e_k(Ck;z,z)\,dV(z) \lesssim \tau_k^{-2}\int_{V\cap b\Omega}(\delta(z',\tau_k))^{-1}\,dS(z'), \tag{5.15}$$

for some positive constant c.

On $\{z \in \Omega \mid d(z) \geq c(\delta(\pi(z),\tau_k))^{1/2}\}$, we know from Proposition 5.1 that

$$\text{tr } e_k(Ck;z,z) \lesssim k(\delta(\pi(z),\tau_k))^{-1} \lesssim k^{1+m}. \tag{5.16}$$

Also, on any relatively compact subset of Ω, we have

$$\text{tr } e_k(Ck;z,z) \lesssim k^2, \tag{5.17}$$

where the constant depends on the compact set (see (5.8)). By definition, we have

$$N_k(Ck) = \int_\Omega \text{tr } e_k(Ck;z,z)\,dV(z). \tag{5.18}$$

It then follows from (5.15)-(5.17) that

$$N_k(Ck) \lesssim k^{1+m}.$$

Note that

$$\lim_{k\to\infty} \frac{\tau_k^{2m}}{\delta(z', \tau_k)} = 0 \tag{5.19}$$

when $z' \in b\Omega$ is of type less than $2m$ and the set of weakly pseudoconvex boundary points has zero surface measure. Combining (5.15)-(5.19), we obtain from the Lebesgue dominated convergence theorem that $\limsup_{k\to\infty} N_k(Ck)/k^{m+1} = 0$ when $m > 1$. □

Remark. Heuristic arguments seem to suggest that the optimal estimates are $N_k(Ck) \lesssim k^m \ln k$ when $m = 2$ and $N_k(Ck) \lesssim k^m$ when $m > 2$.

The remaining subsections are devoted to prove Proposition 5.3. The proof follows along the line of arguments of the proof of Lemma 6.2 in [Fu05b]: we need to show here that the contributions from the curvatures of the metric on the base X and fiber metrics on E^k are negligible. We provide necessary details below.

5.3.2. *Uniform Kohn Estimate.* We will use a slight differently rescaling scheme from the one in the previous section. Following [Fu05b], we flatten the boundary before rescaling the domain and the $\overline{\partial}$-Neumann Laplacian. Let $z' \in b\Omega$. As in Section 5.2, we may choose local holomorphic coordinates (z_1, z_2), centered and orthonormal at z', such that a defining function ρ of $b\Omega$ in a neighborhood $U_{z'}$ of z' has the form given by (5.4). Furthermore, we may assume that the base metric h on X and the fiber metric φ on E are of the forms (5.9) and (5.10) respectively. Write

$$\rho = \text{Re } z_2 + f(z_1) + (\text{Im } z_2)g_1(z_1) + (1/2)(\text{Im } z_2)^2 g_2(z_1) + O(|\text{ Im } z_2|^3),$$

where

$$f(z_1) = P(z_1) + O(|z_1|^{2m+1}), \quad g_1(z_1) = Q(z_1) + O(|z_1|^{m+1}), \quad g_2(z_1) = O(|z_1|).$$

Let

$$(\eta_1, \eta_2) = \Phi_{z'}(z_1, z_2) = (z_1,\ z_2 + h(z_1, \text{Im } z_2) - F(z_1, z_2)),$$

where

$$F(z_1, z_2) = \frac{1}{2} g_2(z_1)(\text{Re } z_2 + h(z_1, \text{Im } z_2))^2 + i(g_1(z_1)(\text{Re } z_2) + g_2(z_1)(\text{Re } z_2)(\text{Im } z_2)).$$

(See [Fu05b], Section 4.)

Let $\hat\rho(z) = \rho(z) - (1/2)g_2(z_1)(\rho(z))^2$. Then $\hat\rho(z)$ is a also defining function for $\Omega_{z'} = \Omega \cap U_{z'}$ near the origin. Let ω_1 and ω_2 be an othonormal basis for $(1,0)$-forms on $U_{z'}$ obtained as in Section 5.2 but with ρ replaced by $\hat\rho$. Let L_1 and L_2 be the dual basis for $T^{1,0}(U_{z'})$.

We now proceed with the rescaling. Write $\delta = \delta(z', \tau)$. For any $\tau > 0$, we define

$$(w_1, w_2) = D_{z',\tau}(\eta_1, \eta_2) = (\eta_1/\tau, \eta_2/\delta).$$

Let $\Phi_{z',\tau} = D_{z',\tau} \circ \Phi_{z'}$ and let $\Omega_{z',\tau} = \Phi_{z',\tau}(\Omega_{z'}) \subset \{(w_1, w_2) \in \mathbb{C}^2 \mid \text{Re } w_2 < 0\}$. (In what follows, we sometimes suppress the subscript z' for economy of notations when this causes no confusions.) Let

$$P_\tau(z') = \{|w_1| < 1,\ |w_2| < \delta^{-1/2}\}.$$

It is easy to see that $R_{C^{-1}\tau}(z') \subset \Phi^{-1}_{z',\tau}(P_\tau(z')) \subset R_{C\tau}(z')$ (see Lemma 4.1 in [Fu05b]). Let $\mathcal{G}_\tau\colon (L^2(\Omega_{z',\tau}))^2 \to L^2_{(0,1)}(\Omega_{z'}, E^k)$ be the transformation defined by

$$\mathcal{G}_\tau(u_1, u_2) = |\det d\Phi_\tau|^{1/2}\big(u_1(\Phi_\tau)\overline{\omega}_1 + u_2(\Phi_\tau)\overline{\omega}_2\big),$$

where on $L^2(\Omega_{z',\tau})$ we use the standard Euclidean metric and we identify as before forms with values in the line bundle E^k with its representation under the given holomorphic trivialization. Let $\tau_k = 1/\sqrt{k}$ and $\delta_k = \delta(z', \tau_k)$ as before. Let

$$Q_{\tau_k}(u, v) = \tau_k^2 Q^k_\Omega(\mathcal{G}_{\tau_k} u, \mathcal{G}_{\tau_k} v) \tag{5.20}$$

be the densely defined, closed sesquilinear form on $(L^2(\Omega_{\tau_k}))^2$ with $\mathcal{D}\,(Q_{\tau_k}) = \{\mathcal{G}^{-1}_{\tau_k}(u);\ u \in \mathcal{D}\,(Q^k_\Omega),\ \text{Supp}\ u \subset U_{z'}\}$. Here, as before, $Q^k_\Omega(\cdot,\cdot)$ is the sesquilinear form associated with the $\overline{\partial}$-Neumann Laplacian $\Box^k_\Omega$ on $L^2_{(0,1)}(\Omega, E^k)$.

The following lemma play a crucial role in the analysis. It is a consequence of Kohn's commutator method and is the analogue of Lemma 4.5 in [Fu05b]. We use $|\!|\!|\cdot|\!|\!|^2_\varepsilon$ to denote the tangential Sobolev norm of order $\varepsilon > 0$ on $\mathbb{C}^2_- = \{(w_1, w_2) \in \mathbb{C}^2 \mid \text{Re}\ w_2 < 0\}$.

Lemma 5.5. *There exists an $\varepsilon > 0$ such that for any sufficiently large k,*

$$Q_{\tau_k}(u, u) + \|u\|^2 \gtrsim |\!|\!|\, u\, |\!|\!|^2_\varepsilon + \tau_k^2\delta_k^{-2}\, |\!|\!|\, \frac{\partial u}{\partial \bar{w}_2}\, |\!|\!|^2_{-1+\varepsilon}, \tag{5.21}$$

for all $u \in \mathcal{D}\,(Q_{\tau_k}) \cap C^\infty_c(P_{\tau_k}(z'))$.

Proof. Let $v = \mathcal{G}_k u = v_1\overline{\omega}_1 + v_2\overline{\omega}_2$. It is easy to see that

$$|k\varphi(z)| \lesssim 1, \qquad |h_{jl}(z) - \delta_{jl}| \lesssim \tau_k$$

on $R_{\tau_k}(z')$. Thus

$$\|u\|^2 \approx \|v\|^2_{h,k\varphi} \approx \|v_1\|^2_0 + \|v_2\|^2_0,$$

where $\|\cdot\|_0$ denotes the L^2-norm corresponding to the standard Euclidean metric on $U_{z'}$ in the (z_1, z_2)-variables. We will also use dV_0 and dS_0 to denote the volume and surface elements in the Euclidean metric.

By integration by parts, we have (see [H65, Ko72]),

$$Q^k_\Omega(v, v) + k\|v\|^2_{h,k\varphi} \gtrsim k\|v\|^2_{h,k\varphi} + \sum_{j,l=1}^{2} \|\overline{L}_j v_l\|^2_{h,k\varphi} + \int_{b\Omega} \big(\partial\overline{\partial}\hat{\rho}(L_1, \overline{L}_1)\big)|v|^2 e^{-k\varphi}\, dS. \tag{5.22}$$

Therefore,

$$\begin{aligned}
Q_{\tau_k}(u, u) + \|u\|^2 &\gtrsim \tau_k^2(Q^k_\Omega(v, v) + k\|v\|^2_{h,k\varphi}) \\
&\gtrsim \tau_k^2\Big(k\|v\|^2_0 + \sum_{j,l=1}^{2} \|\overline{L}_j v_l\|^2_0 + \int_{b\Omega} \big(\partial\overline{\partial}\hat{\rho}(L_1, \overline{L}_1)\big)|v|^2\, dS_0\Big) \\
&\gtrsim \tau_k^2\Big(k\|v\|^2_0 + \sum_{j,l=1}^{2} \|\overline{L}_j v_l\|^2_0 + \sum_{l=1}^{2} \|L_1 v_j\|^2_0\Big) \\
&\gtrsim \|u\|^2 + \sum_{j,l=1}^{2} \|\overline{L}_{j,\tau_k} u_l\|^2 + \sum_{l=1}^{2} \|L_{1,\tau_k} u_l\|^2,
\end{aligned}$$

where $\overline{L}_{j,\tau_k} = \tau_k(\Phi_{z',\tau_k})_*(\overline{L}_j)$. The estimate (5.21) then follows from (the proof of) Lemma 4.5 in [Fu05b]. $\square$

Once Lemma 5.5 is established, the proof of Proposition 5.3 follows along the lines of the proof of Lemma 6.2 in [Fu05b]. Since there are necessary modifications due to the possible present of the $\overline{\partial}$-cohomology, we provide the necessary details for completeness in the next subsection.

5.3.3. *Comparison with an auxiliary Laplacian.* Let ε be the order of the Sobolev norm in Lemma 5.5. Let W_{ε,δ_k} be the space of all $u \in L^2(\mathbb{C}^2_-)$ such that

$$\|u\|^2_{\varepsilon,\delta_k} = \| u \|^2_\varepsilon + \delta_k^{-1} \| \frac{\partial u}{\partial \bar{w}_2} \|^2_{-1+\varepsilon} < \infty. \tag{5.23}$$

Let $\Box_{\varepsilon,\delta_k}$ be the associated densely defined, self-adjoint operator on $L^2(\mathbb{C}^2_-)$ such that $\|u\|^2_{\varepsilon,\delta_k} = \|\Box^{1/2}_{\varepsilon,\delta_k} u\|^2$ and $\mathcal{D}(\Box^{1/2}_{\varepsilon,\delta_k}) = W_{\varepsilon,\delta_k}$. Let N_{ε,δ_k} be its inverse. Let $\chi(w_1, w_2)$ be a smooth cut-off function supported on $\{|w_1| < 2, |w_2| < 2\}$ and identically 1 on $\{|w_1| < 1, |w_2| < 1\}$. Let $\chi_{\delta_k}(w_1, w_2) = \chi(w_1, \delta_k^{1/2} w_2)$. We use $\lambda_j(T)$ to denote the j^{th} singular value (arranged in a decreasing order and repeated according to multiplicity) of a compact operator T. It then follows from the min-max principle that

$$\lambda_{j+k+1}(T_1+T_2) \le \lambda_{j+1}(T_1) + \lambda_{k+1}(T_2) \quad \text{and} \quad \lambda_{j+k+1}(T_3 \circ T_1) \le \lambda_{j+1}(T_1)\lambda_{k+1}(T_3) \tag{5.24}$$

(see [W80]).

For sufficiently large k and j, we then have

$$\lambda_j(\chi_{\delta_k} N^{1/2}_{\varepsilon,\delta_k}) \lesssim (1 + j\delta_k^{1/2})^{-\varepsilon/4} \tag{5.25}$$

(see Lemma 5.3 in [Fu05b]).

Let κ be a cut-off function compactly supported on $U_{z'}$ and identically 1 on a neighborhood of z' of uniform size. Let

$$\mathbf{E}_{\tau_k}(\lambda) = \mathcal{G}^{-1}_{\tau_k} \kappa \mathbf{E}_k(\lambda/\tau_k^2) \kappa \mathcal{G}_{\tau_k} \colon (L^2(\Omega_{\tau_k}))^2 \to (L^2(\Omega_{\tau_k}))^2.$$

The kernel of $\mathbf{E}_{\tau_k}(\lambda)$ is then given by

$$e_{\tau_k}(\lambda; w, w') = e(\lambda/\tau_k^2; \Phi^{-1}_{\tau_k}(w), \Phi^{-1}_{\tau_k}(w'))\kappa(w)\kappa(w')|\det d\Phi^{-1}_{\tau_k}(w)|^{\frac{1}{2}} |\det d\Phi^{-1}_{\tau_k}(w')|^{\frac{1}{2}}. \tag{5.26}$$

We now proceed to prove Proposition 5.3. By (5.26), it suffices to prove that

$$\int_{P_{\tau_k}(z') \cap \Omega_{\tau_k}} \operatorname{tr} \, e_{\tau_k}(C; w, w) \, dV_0(w) \lesssim \delta_k^{-1/2}. \tag{5.27}$$

Let $\Box_{\tau_k} \colon (L^2(\Omega_{\tau_k} \cap P_{\tau_k}(z')))^2 \to (L^2(\Omega_{\tau_k} \cap P_{\tau_k}(z')))^2$ be the operator associated with the sesquilinear form Q_{τ_k} given by (5.20) but with domain

$$\mathcal{D}(Q_{\tau_k}) = \{\mathcal{G}^{-1}_{\tau_k}(u) \mid u \in \mathcal{D}(Q^k_\Omega), \text{Supp } u \subset \Phi^{-1}_{\tau_k}(P_{\tau_k}(z'))\}.$$

Thus $\Box_{\tau_k} = \tau_k^2 \mathcal{G}^{-1}_{\tau_k} \Box \mathcal{G}_{\tau_k}$. Let $N_{\tau_k} = (I + \Box_{\tau_k})^{-1}$. It follows from Lemma 5.5 that

$$Q_{\tau_k}(u, u) + \|u\|^2 \gtrsim \|u\|^2_{\varepsilon,\delta_k},$$

for any $u \in \mathcal{D}(\Box^{1/2}_{\tau_k})$. Therefore,

$$\|u\|^2 = Q_\tau(N^{1/2}_{\tau_k} u, N^{1/2}_{\tau_k} u) + \|N^{1/2}_{\tau_k} u\|^2 \gtrsim \|\Box^{1/2}_{\varepsilon,\delta_k} N^{1/2}_{\tau_k} u\|^2.$$

Thus $N^{1/2}_{\tau_k} = \chi_{\delta_k} N^{1/2}_{\tau_k} = \chi_{\delta_k} N^{1/2}_{\varepsilon,\delta_k} \Box^{1/2}_{\varepsilon,\delta_k} N^{1/2}_{\tau_k}$. It follows from (5.24) and (5.25) that

$$\lambda_j(N^{1/2}_{\tau_k}) \lesssim \lambda_j(\chi_{\delta_k} N^{1/2}_{\varepsilon,\delta_k}) \lesssim (1 + j\delta_k^{1/2})^{-\varepsilon/4}. \tag{5.28}$$

Let K be any positive integer such that $K > 4/\varepsilon$. Let $\chi^{(j)}$, $j = 0, 1, \dots, K$, be a family of cut-off functions supported in $\{|w_1| < 1, |w_2| < 1\}$ such that $\chi^{(0)} = \chi$ and $\chi^{(j+1)} = 1$ on $\operatorname{Supp} \chi^{(j)}$. Let

$$\mathbf{E}^{(l)}_{\tau_k}(\lambda) = \mathcal{G}^{-1}_{\tau_k}\kappa(\tau_k^2\Box^k_\Omega)^l\mathbf{E}_k(\lambda\tau_k^{-2})\kappa\mathcal{G}_{\tau_k} \colon (L^2(\Omega_{\tau_k}))^2 \to (L^2(\Omega_{\tau_k}))^2.$$

Thus $\mathbf{E}^{(0)}_{\tau_k}(\lambda) = \mathbf{E}_{\tau_k}(\lambda)$ and

$$\|\mathbf{E}^{(l)}_{\tau_k}(C)u\| \lesssim \|u\|. \tag{5.29}$$

Furthermore,

$$Q_{\tau_k}(\chi^{(j)}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C)u) = {\tau_k}^2 Q^k_\Omega(\chi^{(j)}_{\delta_k}(\Phi_{\tau_k})({\tau_k}^2\Box^k_\Omega)^l\mathbf{E}(C{\tau_k}^{-2})\kappa\mathcal{G}_{\tau_k}u) \tag{5.30}$$

(Here we use $Q(u)$ to denote $Q(u,u)$ for abbreviation.)

It is straightforward to check that

$$Q^k_\Omega(\theta u, \theta u) = \operatorname{Re}\langle\langle\theta\Box^k_\Omega u, \theta u\rangle\rangle + (1/2)\langle\langle u, [\theta, A]u\rangle\rangle \tag{5.31}$$

where θ is any smooth function on $\overline{\Omega}$ and $A = [\overline{\partial}^*, \theta]\overline{\partial} + \overline{\partial}[\overline{\partial}^*, \theta] + \overline{\partial}^*[\overline{\partial}, \theta] + [\overline{\partial}, \theta]\overline{\partial}^*$. (Here $\overline{\partial}^* = \overline{\partial}^*_{h,k\varphi}$ is the adjoint of $\overline{\partial}$ with respect to the base metric h and fiber metric $k\varphi$.) Note that $[\theta, A]$ is of zero order and its sup-norm is bounded by a constant independent of k.

It follows from (5.30), (5.31), and the Schwarz inequality that for any $u \in (L^2(\Omega_{\tau_k}))^2$,

$$Q_{\tau_k}(\chi^{(l')}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C)u) \lesssim \|\chi^{(l')}_{\delta_k}\mathbf{E}^{(l+1)}_{\tau_k}(C)u\|^2 + \|\chi^{(l'+1)}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C)u\|^2. \tag{5.32}$$

Hence

$$\begin{aligned}\|(I+\Box_{\tau_k})^{1/2}\chi^{(l')}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C)u\|^2 &= Q_{\tau_k}(\chi^{(l')}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C)u) + \|\chi^{(l')}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C)u\|^2\\ &\lesssim \|\chi^{(l')}_{\delta_k}\mathbf{E}^{(l+1)}_{\tau_k}(C)u\|^2 + \|\chi^{(l'+1)}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C)u\|^2 + \|\chi^{(l')}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C)u\|^2\end{aligned}$$

It then follows from (5.24) that

$$\begin{aligned}\lambda_{4j+1}(\chi^{(l')}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C)) &\le \lambda_{j+1}(N^{1/2}_{\tau_k})\lambda_{3j+1}((I+\Box_{\tau_k})^{1/2}_{\tau_k}\chi^{(l')}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C))\\ &\le \lambda_{j+1}(N^{1/2}_{\tau_k})\big(\lambda_{j+1}(\chi^{(l')}_{\delta_k}\mathbf{E}^{(l+1)}_{\tau_k}(C)) + \lambda_{j+1}(\chi^{(l'+1)}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C))\\ &\qquad + \lambda_{j+1}(\chi^{(l')}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C))\big).\end{aligned} \tag{5.33}$$

Using (5.28), (5.29), and(5.33), we then obtain by an inductive argument on $K-(l+l')$ that

$$\lambda_j(\chi^{(l')}_{\delta_k}\mathbf{E}^{(l)}_{\tau_k}(C)) \lesssim (1+j{\delta_k}^{1/2})^{-(K-(l+l'))\varepsilon/4} \tag{5.34}$$

for any pair of non-negative integers l, l' such that $0 \le l+l' \le K$ and for all $j > C_1\delta_k^{-1/2}$, where C_1 is a sufficiently large constant. In particular,

$$\lambda_j(\chi_{\delta_k}\mathbf{E}_{\tau_k}(C)) \lesssim (1+j{\delta_k}^{1/2})^{-K\varepsilon/4}.$$

Since $\mathbf{E}_{\tau_k}(C)$ has uniformly bounded operator norms, we also have that $\lambda_j(\chi_{\delta_k}\mathbf{E}_{\tau_k}(C)) \lesssim 1$. The trace norm of $\chi_{\delta_k}\mathbf{E}_{\tau_k}(C)$ is then given by

$$\sum_{j\lesssim{\delta_k}^{-\frac12}}\lambda_j(\chi_{\delta_k}\mathbf{E}_{\tau_k}(C)) + \sum_{j\gtrsim{\delta_k}^{-\frac12}}\lambda_j(\chi_{\delta_k}\mathbf{E}_{\tau_k}(C)) \lesssim {\delta_k}^{-\frac12} + \sum_{j\gtrsim{\delta_k}^{-\frac12}}(1+j{\delta_k}^{-\frac12})^{-\frac{K\varepsilon}{4}} \lesssim {\delta_k}^{-\frac12}.$$

Inequality (5.27) is now an easy consequence of the above estimate.

5.4. **Estimate of the type.** In this section, we prove the necessity in Theorem 1.3. More precisely, we prove the following:

Proposition 5.6. *Let $\Omega \subset\subset X$ be a smoothly bounded pseudoconvex domain in a complex surface. Let E be holomorphic line bundle over Ω that extends smoothly to $b\Omega$. Let $M > 0$. If for any $C > 0$, there exists $C' > 0$ such that $N_k(Ck) \le C'k^M$ for all sufficiently large integer k, then the type of the domain of $b\Omega$ is $\le 8M$.*

The proof of the above proposition, using a wavelet construction of Lemarié and Meyer [LM86], is a modification of the proof of Theorem 1.3 in [Fu05b]. We provide the full details below. We begin with the following simple well-known consequence of the min-max principle.

Lemma 5.7. *Let Q be a semi-positive, closed, and densely defined sesquilinear form on a Hilbert space. Let $\Box$ be the associated densely defined self-adjoint operator operator. Let $\{u_l \mid 1 \le l \le k\} \subset \mathcal{D}(Q)$. Let λ_l and $\tilde{\lambda}_l$ be the l^{th}-eigenvalues of operator $\Box$ and the hermitian matrix $(Q(u_j, u_l))_{1\le j,l\le k}$ respectively. If*

$$\|\sum_{l=1}^{k} c_l u_l\|^2 \ge C \sum_{l=1}^{k} |c_l|^2$$

for all $c_l \in \mathbb{C}$, then $\tilde{\lambda}_l \ge C\lambda_l$, $1 \le l \le k$.

Proof. By the min-max principle,

$$\tilde{\lambda}_l = \inf\{\tilde{\lambda}(\tilde{L}) \mid \tilde{L} \text{ is an } l\text{-dimensional subspace of } \mathbb{C}^k\}$$

where

$$\tilde{\lambda}(\tilde{L}) = \sup\{\sum_{j,l=1}^{k} c_j \bar{c}_l Q(u_j, u_l) \mid (c_1, \dots, c_k) \in \tilde{L}, \sum_{l=1}^{k} |c_l|^2 = 1\}.$$

Likewise,

$$\lambda_l = \inf_{\substack{L\subset \mathcal{D}(Q)\\ \dim(L)=l}} \sup\{Q(u,u) \mid u \in L, \|u\| = 1\}.$$

For any l-dimensional subspace $\tilde{L}$ of $\mathbb{C}^k$, let $L = \{\sum_{l=1}^{k} c_l u_l \mid (c_1, \dots, c_k) \in \tilde{L}\}$ and let

$$\lambda(L) = \sup\{Q(u,u) \mid u \in L, \|u\| = 1\}.$$

Then $\tilde{\lambda}(\tilde{L}) \ge C_2\lambda(L)$. Hence $\tilde{\lambda}_l \ge C_2\lambda_l$ for all $1 \le l \le k$. $\Box$

Let $z' \in b\Omega$. We follow the notations and setup as in the proof of Proposition 5.1. Suppose the type of $b\Omega$ is $\ge 2m$ at z'. Then $P(z_1) = O(|z_1|^{2m})$. It follows from (5.5) that $Q(z_1) = 0$. Hence

$$\psi(z_1, \mathrm{Im}\, z_2) = O(|z_1|^{2m} + |\mathrm{Im}\, z_2||z_1|^{m+1} + |\mathrm{Im}\, z_2|^2|z_1|). \tag{5.35}$$

Let $b(t)$ be a smooth function supported in $[-1/2,\ 1]$ such that $b(t) = 1$ on $[0,\ 1/2]$ and $b^2(t) + b^2(t-1) = 1$ on $[1/2,\ 1]$. It follows that $\{b(t)e^{2\pi lti} \mid l \in \mathbb{Z}\}$ is an orthogonal system[4] in $L^2(\mathbb{R})$ ([LM86]; see also [Dau88, HG96]). Write $z_2 = s + it$. Let χ be any smooth cut-off function supported on $(-2,\ 2)$ and identically 1 on $(-1, 1)$ and let

$$B(z_2) = (b(t) - ib'(t)s - b''(t)s^2/2)\chi(s/(1 + |t|^2)).$$

[4] In fact, it was shown by Lemarié and Meyer [LM86] that the Fourier transform of $b(t)$ is a wavelet

Then $B(0,t) = b(t)$ and $|\partial B(z_2)/\partial \bar{z}_2| \lesssim |s|^2$. Let $a(z_1)$ be a smooth function identically 1 on $|z_1| \le 1/2$ and supported on the unit disc. For any positive integers j and for any positive integer l such that $2^{mj-1}/j \le l \le 2^{mj}/j$, let

$$u_{j,l}(z) = l^{1/2} 2^{2(m+1)j} a(2^{2j} z_1) B(2^{2mj} z_2) e^{2\pi l 2^{2mj} z_2} e^{k\varphi(z)/2} (g(z))^{-1/2} \overline{\omega}_1,$$

where $g(z) = \det(h_{jl}(z))$. For any sufficiently large j, $u_{j,l}$ is a compactly supported smooth $(0,1)$-form in $\mathcal{D}(Q^k_\Omega)$. (Recall that Q^k_Ω is the sesquilinear form associated with the $\overline{\partial}$-Neumann Laplacian $\square^k_\Omega$ on Ω for $(0,1)$-forms with values in E^k.) Moreover, after the substitutions $(z_1, z_2) \to (2^{-2j} z_1, 2^{-2mj} z_2)$, we have

$$\|u_{j,l}\|^2_{h,k\varphi} = l \int_{\mathbb{C}} |a(z_1)|^2 dV_0(z_1) \int_{\mathbb{R}} dt \int_{-\infty}^{-2^{2mj}\psi(2^{-2j} z_1, 2^{-2mj} t)} |B(z_2)|^2 e^{4\pi l s}\, ds.$$

By (5.35), $|2^{2mj}\psi(2^{-2j} z_1, 2^{-2mj} t)| \lesssim 2^{-2mj}$. Note also that $l 2^{-2mj} \le j^{-1} 2^{-mj}$. Thus

$$\begin{aligned}
\|u_{j,l}\|^2_{h,k\varphi} &\gtrsim l \int_{\mathbb{C}} |a(z_1)|^2 dV_0(z_1) \int_{\mathbb{R}} dt \int_{-\infty}^{-C2^{-2mj}} |B(z_2)|^2 e^{4\pi l s}\, ds \\
&\gtrsim \int_{\mathbb{C}} |a(z_1)|^2 dV_0(z_1) \int_0^{1/2} dt \int_{-1}^{-C2^{-2mj}} l e^{4\pi l s}\, ds \gtrsim 1.
\end{aligned}$$

Similarly, $\|u_{j,l}\|^2_{h,k\varphi} \lesssim 1$, and hence $\|u_{j,l}\|^2_{h,k\varphi} \approx 1$. Also, for any l and l' such that $2^{mj-1}/j \le l, l' \le 2^{mj}/j$,

$$\langle\langle u_{j,l}, u_{j,l'} \rangle\rangle_{h,k\varphi} = \sqrt{ll'} \int_{\mathbb{C}} |a(z_1)|^2 dV_0 \int_{\mathbb{R}} dt \int_{-\infty}^{-2^{2mj}\psi(2^{-2j} z_1, 2^{-2mj} t)} |B(z_2)|^2 e^{2\pi((l+l')s + i(l-l')t)}\, ds$$

We decompose the above integral into two parts. Let A be the above expression with the upper limit in the last integral over s replaced by 0 but keep the lower limit. Let B likewise be the expression with the lower limit replaced by 0 but keep the upper limit. Hence $\langle\langle u_{j,l}, u_{j,l'} \rangle\rangle_{h,k\varphi} = A + B$. We first estimate B:

$$\begin{aligned}
|B| &\le \sqrt{ll'} \int_{\mathbb{C}} |a(z_1)|^2 dV_0 \int_{\mathbb{R}} dt \int_0^{-2^{2mj}\psi(2^{-2j} z_1, 2^{-2mj} t)} |B(z_2)|^2 e^{2\pi(l+l')s}\, ds \\
&\lesssim \frac{\sqrt{ll'}}{l+l'} \left(1 - e^{C(l+l')2^{-2mj}}\right) \lesssim j^{-1} 2^{-mj}.
\end{aligned}$$

To estimate $|A|$, we use the orthogonality of the system of functions $\{b(t) e^{2\pi l t i} \mid l \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$. It follows that if $l \ne l'$, then

$$A = \sqrt{ll'} \int_{\mathbb{C}} |a(z_1)|^2 dV_0 \int_{\mathbb{R}} dt \int_{-\infty}^0 \left(|B(s,t)|^2 - |B(0,t)|^2\right) e^{2\pi((l+l')s + i(l-l')t)}\, ds.$$

Therefore,

$$|A| \lesssim \sqrt{ll'} \int_{\mathbb{C}} |a(z_1)|^2 dV_0 \int_{-1}^1 dt \int_{-\infty}^0 s e^{2\pi(l+l')s}\, ds \lesssim \sqrt{ll'}/(l+l')^2 \lesssim j^{-1} 2^{-mj}.$$

For sufficiently large j and for any k, l such that $2^{mj-1}/j \le l, l' \le 2^{mj}/j$, $l \ne l'$, we then have,

$$|\langle\langle u_{j,l},\ u_{j,l'} \rangle\rangle_{h,k\varphi}| \lesssim j^{-1} 2^{-mj}.$$

For any $c_l \in \mathbb{C}$, we have

$$\begin{aligned}\|\sum_l c_l u_{j,l}\|^2_{h,k\varphi} &= \sum_l |c_l|^2 \|u_{j,l}\|^2_{h,k\varphi} - \sum_{\substack{l,l' \\ l\neq l'}} c_l \bar{c}_{l'} \langle\langle u_{j,l},\ u_{j,l'}\rangle\rangle_{h,k\varphi} \\ &\geq \sum_l |c_l|^2 \|u_{j,l}\|^2_{h,k\varphi} - j^{-1}2^{-mj}\big|\sum_l c_l\big|^2 \\ &\gtrsim (1-j^{-2})\sum_l |c_l|^2 \gtrsim \sum_l |c_l|^2,\end{aligned}$$

where the summations are taken over all integers l between $2^{mj-1}/j$ and $2^{mj}/j$.

Write $L_1^{k\varphi} = e^{k\varphi} L_1 e^{-k\varphi}$. Then

$$Q^k_\Omega(u_{j,l}, u_{j,l}) \lesssim \|u_{j,l}\|^2_{h,k\varphi} + \|\overline{L}_2 u_{j,l}\|^2_{h,k\varphi} + \|L_1^{k\varphi} u_{j,l}\|^2_{h,k\varphi}. \tag{5.36}$$

Recall that

$$\varphi(z) = O(|z|^2) \quad \text{and} \quad |h_{jl}(z) - \delta_{jl}| \lesssim |z|.$$

Moreover,

$$\overline{L}_2 = O(|z|)\frac{\partial}{\partial \bar{z}_1} + O(1)\frac{\partial}{\partial \bar{z}_2}$$

and

$$L_1 = O(1)\frac{\partial}{\partial z_1} + O(|z_1|^{2m-1} + |\operatorname{Im} z_2||z_1|^m + |\operatorname{Im} z_2|^2)\frac{\partial}{\partial z_2}.$$

Let $k = 2^{4j}$. It follows that when $|z_1| \lesssim 2^{-2j}$ and $|z_2| \lesssim 2^{-2mj}$,

$$k|\varphi(z)| \lesssim 1 \quad \text{and} \quad k|\nabla\varphi(z)| \lesssim 2^{2j}.$$

Therefore, on the one hand,

$$\begin{aligned}\|\overline{L}_2 u_{j,l}\|^2_{h,k\varphi} &\lesssim \||z|\frac{\partial u_{j,l}}{\partial \bar{z}_1}\|^2_{h,k\varphi} + \|\frac{\partial u_{j,l}}{\partial \bar{z}_2}\|^2_{h,k\varphi} \\ &\lesssim 2^{4j} + 2^{4mj}\int_{\mathbb{C}} |a(z_1)|^2\, dV_0 \int_{-1}^1 dt \int_{-\infty}^{-2^{2mj}\psi(2^{-2j}z_1, 2^{-2mj}t)} l\Big|\frac{\partial B}{\partial \bar{z}_2}\Big|^2 e^{4\pi l s}\, ds \\ &\lesssim 2^{4j} + 2^{4mj}\int_{\mathbb{C}} |a(z_1)|^2\, dV_0 \int_{-1}^1 dt \int_{-\infty}^{C2^{-2mj}} l s^4 e^{4\pi l s}\, ds \\ &\lesssim 2^{4j} + 2^{4mj} l^{-4} \lesssim 2^{4j}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|L_1^{k\varphi} u_{j,l}\|^2_{h,k\varphi} &\lesssim \|(k\nabla\varphi)u_{j,l}\|^2_{h,k\varphi} + \Big\|\frac{\partial u_{j,l}}{\partial z_1}\Big\|^2_{h,k\varphi} + \Big\|(|z_1|^{2m-1} + |t||z_1|^m + t^2)\frac{\partial u_{j,l}}{\partial z_2}\Big\|^2_{h,k\varphi} \\ &\lesssim 2^{4j} + 2^{-4mj+4j}\int_{\mathbb{C}} |a(z_1)|^2 dV_0 \int_{-1}^1 dt \int_{-\infty}^{C2^{-2mj}} l^3 e^{4\pi l s}\, ds \\ &\lesssim 2^{4j}.\end{aligned}$$

We thus have

$$Q^k_\Omega(u_{j,l}, u_{j,l}) \lesssim 2^{4j}. \tag{5.37}$$

Let $\lambda_{k,l}$ be the l^{th} eigenvalues of $\Box^k_\Omega$. The hypothesis of Proposition 5.6 implies that

$$\lambda_{k,l} \geq Ck \tag{5.38}$$

when $l > C'k^M$. Proving by contradiction, we suppose that $M < m/4$. It follows from Lemma 5.7 and (5.38) that,

$$\sum_{l=j^{-1}2^{mj-1}}^{j^{-1}2^{mj}} Q_\Omega^k(u_{j,l}, u_{j,l}) \geq \sum_{l=1}^{j^{-1}2^{mj-1}} \lambda_{k,l} \geq C(j^{-1}2^{mj-1} - C'k^M)k. \tag{5.39}$$

Combining (5.37) and (5.39), we then have

$$j^{-1}2^{mj-1}2^{4j} \gtrsim C(j^{-1}2^{mj-1} - C'k^M)k.$$

Dividing both sides by $k = 2^{4j}$ and $j^{-1}2^{mj-1}$, we obtain

$$1 \gtrsim C(1 - C'2^{(4M-m)j+1}).$$

Since by assumption, C can be chosen arbitrarily large, we arrive at a contradiction by letting $j \to \infty$. We thus conclude the proof of Proposition 5.6.

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