b-COMPLETION OF PSEUDOHERMITIAN MANIFOLDS

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Abstract. We study the interrelation among pseudohermitian and Lorentzian geometry as prompted by the existence of the Fefferman metric. Specifically for any nondegenerate CR manifold M we build its b-boundary $\dot{M}$. This arises as a S^1 quotient of the b-boundary of the (total space of the canonical circle bundle over M endowed with the) Fefferman metric. Points of $\dot{M}$ are shown to be endpoints of b-incomplete curves. A class of inextensible integral curves of the Reeb vector on a pseudo-Einstein manifold is shown to have an endpoint on the b-boundary provided that the horizontal gradient of the pseudohermitian scalar curvature satisfies an appropriate boundedness condition.

Dedicated to the memory of Stere Ianuş[4]

1. Introduction

The present paper is part of a larger programme aiming to the study of the relationship among space-time physics and Cauchy-Riemann geometry. A *space-time* is a connected C^∞ Hausdorff manifold $\mathfrak{M}$ of dimension $m \geq 2$ which has a countable basis and is equipped with a Lorentzian metric F of signature $(- + \cdots +)$ and a time orientation.

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[4]Romanian mathematician († 2010). S. Ianuş had a life-long interest in general relativity theory [from a differential geometric viewpoint - starting with his Ph.D. thesis [31] written under Ghe. Vrânceanu († 1979)] and CR geometry. Among his many contributions S. Ianuş authored the excellent monograph [32].

The metric F isn't positive definite yet it furnishes a distinction of tangent vectors into types (timelike, null, spacelike) leading to a natural *causality theory* on $\mathfrak{M}$ (cf. [5], p. 21-32). A CR structure is a bundle theoretic recast of the tangential Cauchy-Riemann equations i.e. given an orientable connected $(2n+1)$-dimensional C^∞ manifold M a *CR structure* (of CR dimension n) is a complex subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$, of complex rank n, such that i) $T_{1,0}(M) \cap \overline{T_{1,0}(M)} = (0)$ and ii) if $Z, W \in C^\infty(T_{1,0}(M))$ then $[Z, W] \in C^\infty(T_{1,0}(M))$ (cf. [19], p. 3). The *tangential Cauchy-Riemann* operator is the first order differential operator $\overline{\partial}_b : C^1(M) \to C(T_{0,1}(M)^*)$ given by $\left(\overline{\partial}_b f\right) \overline{Z} = \overline{Z}(f)$ and a C^1 solution to $\overline{\partial}_b f = 0$ is a *CR function.* CR manifolds (i.e. manifolds endowed with CR structures) appear mainly as real hypersurfaces in complex manifolds although nonembeddable examples exist. The *embedding problem* is then to look for an immersion of the given CR manifold $(M, T_{1,0}(M))$ into a complex manifold V such that the CR structure be induced by the complex structure of V i.e. $T_{1,0}(M) = [T(M) \otimes \mathbb{C}] \cap T^{1,0}(V)$ (where $T^{1,0}(V)$ is the holomorphic tangent bundle over V). If M is embedded in V then any holomorphic function on V defined in a neighborhood of M restricts to a CR function on M and the *CR extension problem* is to decide whether the restriction morphism $\mathcal{O}(V) \to \mathrm{CR}^1(M)$ is surjective. The embedding and CR extension problems are related, both have local and global aspects, and both are physically meaningful (cf. [46] and [58]). The geometric approach to the study of tangential Cauchy-Riemann equations is through the use of *pseudohermitian structures* (as introduced by S.M. Webster, [57]). As a mere consequence of orientability the conormal bundle $H(M)^\perp_x = \{\omega \in T^*_x(M) : \mathrm{Ker}(\omega) \supset H(M)_x\}$, $x \in M$, is an oriented real line bundle (over a connected manifold) hence $H(M)^\perp \approx M \times \mathbb{R}$ (a bundle isomorphism). Here $H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ is the *Levi distribution.* Hence globally defined nowhere zero sections $\theta \in C^\infty(H(M)^\perp)$ exist and a synthetic object $(M, T_{1,0}(M), \theta)$ is a *pseudohermitian manifold.* The terminology (cf. [57]) is motivated by the formal similarity to Hermitian geometry i.e. under the assumption of *nondegeneracy* on any pseudohermitian manifold one may build (cf. [57], [56]) a unique linear connection ∇ (the *Tanaka-Webster connection*) resembling Chern's connection of a Hermitian manifold (cf. e.g. [59]). The relationship to semi-Riemannian geometry is due the presence of a semi-Riemannian metric F_θ on the total bundle of the canonical circle bundle $S^1 \to C(M) \to M$ which transforms conformally under a change of θ and which may be explicitly computed in terms of pseudohermitian invariants (such as the connection 1-forms of ∇ and their derivatives, the pseudohermitian scalar curvature, etc.). Also F_θ is a Lorentz metric when M is strictly pseudoconvex and the Lorentzian manifold $(C(M), F_\theta)$ admits a natural time-orientation so that $C(M)$ is a space-time. As it turns out, analysis and geometry problems on $C(M)$ and M are related e.g. the *CR Yamabe problem* (find $u \in C^\infty(M)$ such that the Tanaka-Webster connection of $e^u\theta$ has constant pseudohermitian scalar curvature, cf. [19]) is precisely the Yamabe problem for the Fefferman metric F_θ.

The scope of the present paper is to exploit B.G. Schmidt's construction (cf. [50]) of a b-boundary for the space-time $(C(M), F_\theta)$ in order to build a b-completion $\overline{M}$ and b-boundary $\dot{M}$ of the given CR manifold M. The completion $\overline{M}$ is built in § 4 and Theorem 1 there lists its main topological properties. It should be emphasized that the construction of the b-boundary and b-completion depend on a fixed contact form θ and the resulting objects $\overline{M}$ and $\dot{M}$ are not CR invariants. In § 5 we take up the problem of differential geometric conditions (in terms of pseuohermitian invariants) on a smooth curve in M implying that its endpoint lies on the b-boundary $\dot{M}$ (cf. Theorem 3). The main ingredient is an acceleration condition in [15], the Dodson-Sulley-Williamson lemma, of which a rigorous statement and precise proof are given in Appendix A to this paper.

The constructions in § 4 are actually sufficiently general to carry over easily from the case of a strictly pseudoconvex CR manifold to that of a nondegenerate CR manifold of arbitrary signature (r, s). To preserve a solid bond to physics of space-times we only detail the constructions (of b-completions and b-boundaries) for strictly pseudoconvex CR manifolds M (whose total space of the canonical circle bundle is a space-time, see § 3 below) yet a brief application is given when $M = \mathbb{P}(\mathbb{T}_0) \setminus I$ is Penrose's twistor CR manifold (a 5-dimensional nondegenerate CR manifold of signature $(+-)$, cf. [46]) separating right-handed and left-handed spinning photons.

The problem of building a CR analog $\partial_{\rm CR} M$ to the conformal boundary $\partial_c \mathfrak{M}$ of a given space-time (cf. G.B. Schmidt, [52]) may be solved along the lines in § 4 and the solution will be presented in a further paper. Since the restricted conformal class of the Fefferman metric is a CR invariant, $\partial_{\rm CR} M$ would be a new CR invariant. A direct construction of $\partial_{\rm CR} M$ (avoiding the use of the Fefferman metric) is feasible (as suggested by the Reviewer) by a Cartan geometry approach (this problem will be addressed elsewhere). Aside from the attempts due to B. Bosshard, [9], and R.A. Johnson, [34] (partially confined to toy 2-dimensional models) no explicit calculations of b-boundaries of space-times seem to be available in the present day literature. As well as in the classical theory (as built in [50]-[52]) there is a lack of explicit computability of the b-boundary of a CR manifold yet emerging new approaches and techniques (cf. A.M. Amores & M. Gutiérrez, [1], H. Friedrich, [24], M. Gutiérrez, [28], F. Stahl, [53]) are rather promising and (as opposed to more pessimistic expectations, cf. R.K. Sachs, [49], p. 220) b-boundary techniques may play a strong role in general relativity.

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2. A brief review of CR geometry

2.1. **CR structures, Levi form, Webster metric.** Let $(M, T_{1,0}(M))$ be a nondegenerate CR manifold, of CR dimension n. Then every pseudohermitian structure θ on M is a contact form i.e. $\theta \wedge (d\theta)^n$ is a volume form. Once nondegeneracy is assumed, the *Reeb vector* is the unique globally defined everywhere nonzero tangent vector field $T \in \mathfrak{X}(M)$, transverse to the Levi distribution, determined by $\theta(T) = 1$ and $T \rfloor d\theta = 0$. The *Levi form* is

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(M).$$

Let $J : H(M) \to H(M)$ be the complex structure along the Levi distribution i.e. $J(Z + \overline{Z}) = i(Z - \overline{Z})$ for any $Z \in T_{1,0}(M)$. It is customary to consider also the *real* Levi form i.e.

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M).$$

Then G_θ is bilinear, symmetric and compatible with J (as a mere consequence of the integrability conditions imposed on $T_{1,0}(M)$) and the $\mathbb{C}$-linear extension of G_θ to $H(M) \otimes \mathbb{C}$ coincides with L_θ on $T_{1,0}(M) \otimes T_{0,1}(M)$. When M is nondegenerate (an assumption we shall maintain for the remainder of this paper) there exist nonnegative integers $r, s \in \mathbb{Z}_+$ (with $r + s = n$) such that $L_{\theta,x}$ has (constant) signature (r, s) at any point $x \in M$. Under a transformation of contact form $\hat{\theta} = \lambda\,\theta$ (where $\lambda : M \to \mathbb{R} \setminus \{0\}$ is a C^∞ function) the Levi form changes as $L_{\hat{\theta}} = \lambda L_\theta$ (hence the analogy among CR and conformal geometry). In particular the pair (r, s) is a CR invariant (referred to as the *signature* of the CR manifold M). The real Levi form G_θ has signature $(2r, 2s)$. As $T(M) = H(M) \oplus \mathbb{R}T$ the real Levi form G_θ admits a natural contraction g_θ given by

$$g_\theta(X,Y) = G_\theta(X,Y), \quad g_\theta(X,T) = 0, \quad g_\theta(T,T) = 1,$$

for any $X, Y \in H(M)$. Then g_θ (the *Webster metric* of (M, θ)) is a semi-Riemannian metric of signature $(2r+1, 2s)$. When M is strictly pseudoconvex (i.e. L_θ is positive definite for some θ) $(M, H(M), G_\theta)$ is a sub-Riemannian manifold (in the sense of [55]) and g_θ is a Riemannian metric on M. In particular M admits two natural distance functions $d, d_H : M \times M \to [0, +\infty)$ where d is associated to the Webster metric (cf. e.g. [37], Vol. I, p. 157-158) while d_H is the Carnot-Carathéodory metric associated to the sub-Riemannian structure $(H(M), G_\theta)$. The Carnot-Charathéodory distance among two points is measured by employing curves tangent to $H(M)$ only hence $d(x,y) \le d_H(x,y)$ for any $x, y \in M$ (thus justifying the use of the term *contraction* in the description of the Webster metric).

2.2. **Horizontal gradients, Tanaka-Webster connection.** The *horizontal gradient* of a function $u \in C^1(M)$ is given by $\nabla^H u = \Pi_H \nabla u$ where $\Pi_H : T(M) \to H(M)$ is the projection relative to the direct sum decomposition $T(M) = H(M) \oplus \mathbb{R}T$ and $g_\theta(\nabla u, X) = X(u)$ for any $X \in \mathfrak{X}(M)$. The *Tanaka-Webster connection* is the unique linear connection ∇ on (M, θ) satisfying i) $H(M)$ is ∇-parallel, ii) $\nabla g_\theta = 0$ and $\nabla J = 0$, iii) the torsion tensor field T_∇ is *pure* i.e.

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2i L_\theta(Z, \overline{W}) T,$$

for any $Z, W \in T_{1,0}(M)$ and $\tau \circ J + J \circ \tau = 0$. The same symbol J denotes the extension of $J : H(M) \to H(M)$ to an endomorphism of the tangent bundle by requiring that $JT = 0$. Also $\tau(X) = T_\nabla(T, X)$ for any $X \in \mathfrak{X}(M)$ (τ is the *pseudohermitian torsion* of ∇). The divergence operator $\operatorname{div} : \mathfrak{X}(M) \to C^\infty(M)$ is meant with respect to the volume form $\Psi_\theta = \theta \wedge (d\theta)^n$ i.e. $\mathcal{L}_X \Psi_\theta = \operatorname{div}(X)\, \Psi_\theta$ for any $X \in \mathfrak{X}(M)$ where $\mathcal{L}$ denotes Lie derivative. The *sublaplacian* is the formally self-adjoint, second order, degenerate elliptic operator Δ_b given by $\Delta_b u = -\operatorname{div}\left(\nabla^H u\right)$ for any $u \in C^2(M)$. For further use we set $A(X,Y) = g_\theta(X, \tau Y)$ for any $X, Y \in \mathfrak{X}(M)$. By a result of S.M. Webster, [57], A is symmetric.

Example 1. (*Siegel-Fefferman domains*) For each $\delta \ge 0$ let $\rho_\delta(z,w) = \operatorname{Im}(w) - |z|^2 - \delta \operatorname{Re}(w)\, |z|^4$. We consider the family of domains $\Omega_\delta = \{(z,w) \in \mathbb{C}^2 : \rho_\delta(z,w) > 0\}$ so that Ω_0 is the *Siegel domain* while Ω_1

was introduced in [22] (cf. also [10], p. 164). Each boundary $\partial\Omega_\delta$ is a CR manifold, of CR dimension 1, with the CR structure

$$T_{1,0}(\partial\Omega_\delta) = [T(\partial\Omega_\delta) \otimes \mathbb{C}] \cap T^{1,0}(\mathbb{C}^2)$$

induced by the complex structure of $\mathbb{C}^2$. A (global) frame of $T_{1,0}(\partial\Omega_\delta)$ is $Z = \partial/\partial z - 2\,\overline{z}\,F_\delta\,\partial/\partial w$ where $F_\delta(z,w) = [1 + \delta|z|^2(w+\overline{w})]\,[\delta|z|^4 + i]^{-1}$ hence the Levi form is

$$g_{1\overline{1}} = \partial\overline{\partial}\rho_\delta\,(Z,\overline{Z}) = \frac{1}{2} + \frac{\delta^2|z|^8 - 1}{\delta^2|z|^8 + 1}\left[1 + \delta|z|^2(w+\overline{w})\right].$$

Let $L_\delta = \{(z,w) \in \partial\Omega_\delta : g_{1\overline{1}} = 0\}$ (the null locus of the Levi form) so that $L_0 = \emptyset$ and $L_\delta \approx \mathbb{C} \setminus S^1\left(\delta^{-1/4}\right)$ for any $\delta > 0$. Here $S^1(r) \subset \mathbb{C}$ is the circle of radius r and center the origin. Indeed let $\ell_\delta \subset \mathbb{C}$ be the line of equation $v = u + \delta^{-1/2}$. Then L_δ consists of all $(z, u+iv) \in \partial\Omega_\delta \setminus \left[S^1\left(\delta^{-1/4}\right) \times \ell_\delta\right]$ such that

$$u = \frac{1}{2\delta|z|^2}\left[\frac{1}{2}\,\varphi_\delta(|z|) - 1\right], \quad v = \frac{|z|^2}{2}\left[\frac{1}{2}\,\varphi_\delta(|z|) + 1\right],$$

and $\varphi_\delta(r) = (\delta^2 r^8 + 1)\,/\,(\delta^2 r^8 - 1)$ for any $r \in [0,+\infty) \setminus \left\{\delta^{-1/4}\right\}$. Next $M_\delta = \partial\Omega_\delta \setminus L_\delta$ is a strictly pseudoconvex CR manifold with two connected components $M_\delta^{\pm}$ on which the Levi form L_{θ_δ} is respectively positive and negative definite (here $\theta_\delta = \frac{i}{2}(\overline{\partial} - \partial)\rho_\delta$). □

Let R^∇ be the curvature tensor field of the Tanaka-Webster connection of (M,θ) and let us set

$$\mathrm{Ric}(X,Y) = \mathrm{trace}\left\{Z \in T(M) \mapsto R^\nabla(Z,Y)X\right\}.$$

If $\{T_\alpha : 1 \le \alpha \le n\}$ is a local frame of $T_{1,0}(M)$ then $R_{\alpha\overline{\beta}} = \mathrm{Ric}(T_\alpha, T_{\overline{\beta}})$ is the *pseudohermitian Ricci* tensor. We also set $g_{\alpha\overline{\beta}} = L_\theta(T_\alpha\,,\,T_{\overline{\beta}})$ (the local coefficients of the Levi form) and $[g^{\alpha\overline{\beta}}] = [g_{\alpha\overline{\beta}}]^{-1}$. The *pseudohermitian scalar curvature* is $\rho = g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$. A nondegenerate pseudohermitian manifold (M,θ) is (globally) *pseudo-Einstein* if $R_{\alpha\overline{\beta}} = (\rho/n)\,g_{\alpha\overline{\beta}}$. The sphere S^{2n+1} (carrying the standard contact form, cf. e.g. [19], p. 60) is pseudo-Einstein.

Example 2. (*Grauert tubes*) Let (V,g) be a compact connected C^ω Riemannian manifold and $T^{*\,\epsilon}V = \{\xi \in T^*(V) : g^*(\xi,\xi)^{1/2} < \epsilon\}$. There is $\epsilon_0 > 0$ such that $T^{*\,\epsilon_0}V$ admits a canonical complex structure and $M_\epsilon = \partial\,T^{*\,\epsilon}V$ is a strictly pseudoconvex CR manifold for every $0 < \epsilon \le \epsilon_0$ (cf. [26]). Let $\phi(v) = |v|_g^2$ be the squared g-length function and $\theta_\epsilon = \iota_\epsilon^*\left(-\mathrm{Im}\,\overline{\partial}\phi\right)$ where ι_ϵ is the inclusion. If V is a harmonic manifold (in the sense of [7]) then each $(M_\epsilon\,,\,\theta_\epsilon)$ is pseudo-Einstein (cf. [54], p. 394). □

Let $M \subset \mathbb{C}^2$ be a nondegenerate real hypersurface. A curve γ in M is a *chain* if for each point $p \in \gamma$ there is an open set $U \subset M$ and a local biholomorphism $\Phi : \Omega \subset \mathbb{C}^2 \to \Phi(\Omega) \subset \mathbb{C}^2$ defined on an open set $\Omega \supset U$ such that $p \in U$ and

$$\Phi(U) = \{(z, u + iv) \in \Phi(\Omega) : v = |z|^2 + \sum_{k,j \geq 2} F_{kj}(u) z^j \overline{z}^k\},$$

for some functions $F_{kj}(u)$ such that $F_{32}(u) = 0$ and $\Phi(\gamma \cap U)$ lies on the u-axis (cf. e.g. [33], p. 85). The family of chains is a CR invariant. Chains on $\partial\Omega_0 = \{(z, u + iv) \in \mathbb{C}^2 : v = |z|^2\}$ are the intersections of $\partial\Omega_0$ with complex lines (cf. Theorem 2 in [33], p. 85). By a result of C. Fefferman, [22], there is an infinite family of chains on M_1^- (the connected component of $\partial\Omega_1 \setminus L_1$ containing the origin) which spiral to the origin and the origin is the only spiral point on M_1^- (cf. also Theorem 4.15 in [10], p. 164).

3. Fefferman space-times

3.1. **Canonical bundle, Fefferman metric.** Let $(M, T_{1,0}(M))$ be a CR manifold of CR dimension n. A complex valued differential p-form ω on M is a *$(p, 0)$-form* if $T_{0,1}(M) \rfloor \omega = 0$. Let $\Lambda^{p,0}(M) \to M$ be the relevant bundle i.e. C^∞ sections in $\Lambda^{p,0}(M)$ are the $(p, 0)$-forms. Let $\mathbb{R}^+ = (0, +\infty)$ be the multiplicative positive reals and $C(M) = [\Lambda^{n+1,0}(M) \setminus (0)] / \mathbb{R}^+$. Then $C(M)$ is the total space of a principal S^1-bundle $\pi : C(M) \to M$ (the *canonical circle bundle*, cf. e.g. [19], p. 119). From now on we assume that M is nondegenerate of signature (r, s). For each contact form θ on M there is a semi-Riemannian metric F_θ on $C(M)$ (the *Fefferman metric* of (M, θ)) of signature $(2r+1, 2s+1)$ expressed by

$$F_\theta = \pi^* \tilde{G}_\theta + 2(\pi^*\theta) \odot \sigma \tag{1}$$

where $\sigma \in C^\infty(T^*(C(M)))$ is a connection 1-form in $S^1 \to C(M) \to M$ determined by the contact form θ (cf. (2.31) in [19], p. 129, and (2.8) in [27], p. 857). Also $\tilde{G}_\theta$ is the extension of G_θ to $T(M)$ got by requiring that $\tilde{G}_\theta(T, X) = 0$ for any $X \in \mathfrak{X}(M)$. Throughout we adopt the notations and conventions in [41]. However a review of the approach in [41] (or [19], p. 122-131) shows that strict pseudoconvexity of M as required in [41] may be relaxed to nondegeneracy. The connection form σ may be explicitly computed in terms of pseudohermitian invariants (cf. [41])

$$\sigma = \frac{1}{n+2} \left\{ d\gamma + \pi^* \left(i\, \omega_\alpha{}^\alpha - \frac{i}{2}\, g^{\alpha\overline{\beta}}\, dg_{\alpha\overline{\beta}} - \frac{\rho}{4(n+1)}\, \theta \right) \right\}. \tag{2}$$

Here $\gamma : \pi^{-1}(U) \to \mathbb{R}$ is a local fibre coordinate on $C(M)$ i.e. given a local frame $\{T_\alpha : 1 \le \alpha \le n\}$ of $T_{1,0}(M)$ defined on the open set $U \subset M$ let $\{\theta^\alpha : 1 \le \alpha \le n\}$ be the corresponding adapted coframe; if $c = [\omega] \in \pi^{-1}(U)$ (brackets indicate classes mod S^1) then $\omega = \lambda\,(\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n)_x$ for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $\gamma(c) = \arg(\lambda/|\lambda|)$ where $\arg : S^1 \to [0, 2\pi)$.

Let us set $\mathfrak{M} = C(M)$ for simplicity. Let $S \in \mathfrak{X}(\mathfrak{M})$ be the tangent to the S^1 action i.e. locally $S = [(n+2)/2]\,\partial/\partial\gamma$. Then $F_\theta(S,S) = 0$ i.e. S is null (or lightlike). By a result in [27] $\mathcal{L}_S F_\theta = 0$ (S is a Killing vector field). Also $\mathrm{Ric}_{F_\theta}(S,S) = n/2$ and $S \rfloor W_{F_\theta} = S \rfloor C_{F_\theta} = 0$ where Ric_{F_θ}, W_{F_θ} and C_{F_θ} are respectively the Ricci, Weyl and Cotton tensor fields of $(\mathfrak{M}, F_\theta)$. Viceversa (again by a result in [27]) *any* semi-Riemannian metric F of signature $(2r+1, 2s+1)$ on a manifold $\mathfrak{M}$ may be realized locally as the Fefferman metric associated to some contact form on the (locally defined) quotient $M = \mathfrak{M}/S$ provided that F admits a null Killing vector field S such that $\mathrm{Ric}_F(S,S) > 0$ and $S \rfloor W_F = S \rfloor C_F = 0$ (cf. Theorem 3.1 in [27], p. 860). An obstruction to the global statement may be pinned down as a cohomology class in $H^1(M, S^1)$ (when $\mathfrak{M}$ is the total space of a principal S^1-bundle over a $(2n+1)$-dimensional manifold M, cf. Theorem 4.1 in [27], p. 872). By a result in [41] none of the Fefferman metrics

$$F_{e^f\theta} = e^{f\circ\pi} F_\theta\,, \quad f \in C^\infty(M), \tag{3}$$

is Einstein yet (by a result in [43]) if θ is pseudo-Einstein and transversally symmetric then F_θ is locally conformal to an Einstein metric (however the conformal factor depends on the local fibre coordinate). Equality (3) holds by Theorem 2.3 in [19], p. 128. In particular the *restricted* conformal class $[F_\theta] = \{e^{f\circ\pi} F_\theta : f \in C^\infty(M)\}$ is a CR invariant.

3.2. **Causality theory.** Let M be strictly pseudoconvex i.e. the Levi form G_θ is positive definite ($s = 0$) for some θ. Let $T^\uparrow \in \mathfrak{X}(\mathfrak{M})$ be the horizontal lift of the Reeb vector with respect to σ i.e. $T^\uparrow_c \in \mathrm{Ker}(\sigma)_c$ and $(d_c\pi)T^\uparrow_c = T_{\pi(c)}$ for any $c \in \mathfrak{M}$. The tangent vector field $T^\uparrow - S$ is timelike hence the Lorentzian manifold $(\mathfrak{M}, F_\theta)$ is time-oriented. Therefore $\mathfrak{M}$ is a space-time, referred to hereafter as the *Fefferman space-time*. As to causality theory on the space-time $(\mathfrak{M},\, F_\theta\,,\, T^\uparrow - S)$ one adopts the conventions in [5]. Given $c, c' \in \mathfrak{M}$ we write $c << c'$ (respectively $c \le c'$) if there is a smooth future-directed timelike curve (respectively if either $c = c'$ or there is a future-directed nonspacelike curve) from c to c'. The *chronological future/past* (respectively *causal future/past*) of $c \in \mathfrak{M}$ is denoted by $I^\pm(c)$ (respectively $J^\pm(c)$) and

$$I^+(c) = \{c' \in \mathfrak{M} : c << c'\}, \quad I^-(c) = \{c' \in \mathfrak{M} : c \in I^+(c')\},$$

$$J^+(c) = \{c' \in \mathfrak{M} : c \leq c'\}, \quad J^-(c) = \{c' \in \mathfrak{M} : c \in J^+(c')\}.$$

The subsets $I^\pm(c) \subset \mathfrak{M}$ are known to be open (while $J^\pm(c)$ are neither open nor closed, in general). The space-time $\mathfrak{M}$ is *chronological* (respectively *causal*) if $c \notin I^+(c)$ (respectively if $c \notin J^+(c)$) for any $c \in \mathfrak{M}$. If M is compact (e.g. $M = S^{2n+1} \subset \mathbb{C}^{n+1}$) then $\mathfrak{M}$ is neither causal nor chronological. Indeed if this the case then $\mathfrak{M}$ is compact hence (by Proposition 2.6 in [5], p. 23) $\mathfrak{M}$ contains a closed timelike curve.

Let $\alpha : [a, b] \to \mathfrak{M}$ be a curve in $\mathfrak{M}$. A point $c \in \mathfrak{M}$ is the *endpoint* of α corresponding[5] to $t = b$ if $\lim_{t \to b^-} \alpha(t) = c$. If $\alpha : [a, b] \to \mathfrak{M}$ is a future (respectively past) directed nonspacelike curve with endpoint c corresponding to $t = b$ then the point c is a *future* (respectively *past*) *endpoint* of α. A nonspacelike curve in $\mathfrak{M}$ is *future* (respectively *past*) *inextensible* if it has no future (respectively past) endpoint. Given a space-time $\mathfrak{N}$ a *Cauchy surface* is a subset $\Sigma \subset \mathfrak{N}$ such that every inextensible nonspacelike curve intersects Σ exactly once. Moreover $\mathfrak{N}$ is *globally hyperbolic* if the intersection of causal future and past of arbitrary points is a compact set. The *Alexandrov topology* on $\mathfrak{M}$ is the topology $\mathcal{A}_\mathfrak{M}$ generated by the basis of open sets $\{I^+(c) \cap I^-(c') : c, c' \in \mathfrak{M}\}$. The *Alexandrov topology* on M is the topology $\mathcal{A}_M$ consisting of all sets $U \subset M$ such that $\pi^{-1}(U) \in \mathcal{A}_\mathfrak{M}$.

Proposition 1.
i) *The Fefferman space-time* $\mathfrak{M}$ *admits closed null curves and hence* $\mathfrak{M}$ *is not causal.* A fortiori $\mathfrak{M}$ *cannot be distinguishing, strongly causal, stably causal, causally continuous, causally simple or globally hyperbolic. In particular* $\mathfrak{M}$ *admits no Cauchy surface.*
ii) *The chronological and causal future/past maps* $c \in \mathfrak{M} \mapsto I^\pm(c) \subset \mathfrak{M}$ *and* $c \in \mathfrak{M} \mapsto J^\pm(c) \subset \mathfrak{M}$ *are constant on the fibres of* $\pi : \mathfrak{M} \to M$.
iii) *The Alexandrov topology* $\mathcal{A}_\mathfrak{M}$ *doesn't agree with the topology of* $\mathfrak{M}$ *as a manifold. The Alexandrov topology* $\mathcal{A}_M$ *is strictly contained in the quotient topology.*

Proof. i) Let $c \in \mathfrak{M}$ with $\pi(c) = x \in M$. Then $\alpha : [0, 1] \to \mathfrak{M}$, $\alpha(t) = e^{2\pi i t} c$, $0 \leq t \leq 1$, is a smooth closed null curve in $\mathfrak{M}$ hence $\mathfrak{M}$ is not causal. Actually α is contained in the fibre $\pi^{-1}(x)$ hence it is a (closed) null geodesic of $\mathfrak{M}$. The listed features of $\mathfrak{M}$ imply one another (in reversed order, cf. e.g. [5], p. 32) and all imply causality. Inexistence of Cauchy surfaces in $\mathfrak{M}$ then follows by the classical characterization of global hyperbolicity in [29], p. 211-212.

[5]The question whether the definition should be formulated with an interval of the form $[a, b]$ or $[a, b)$ is of course immaterial (the existence of $\lim_{t \to b^-} \alpha(t)$ depends on the topology of $\mathfrak{M}$ whether $\alpha(b)$ is defined or not).

ii) Let $c, c' \in \pi^{-1}(x)$ and $c'' \in I^+(c)$. The circle action is transitive along the fibres hence $c' = e^{i\varphi}c$ for some (unique) $\varphi \in [0, 2\pi)$. Let $\beta : [0,1] \to \mathfrak{M}$ be given by $\beta(t) = e^{it\varphi}c$ for any $0 \leq t \leq 1$. Let $0 < t_0 < 1$ and (U, x^A) a local coordinate system on M such that $(t_0 - \delta, t_0 + \delta) \subset [0,1]$ and $\beta(t) \in \pi^{-1}(U)$ for all $|t - t_0| < \delta$ and some $\delta > 0$. We may assume that U also carries an adapted coframe $\{\theta^\alpha : 1 \leq \alpha \leq n\}$. If $c = [\lambda\ (\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n)_x]$ then $x^A(\beta(t)) = x^A(x)$ and $\gamma(\beta(t)) = \arg(\lambda/|\lambda|) + t\varphi + 2\pi N(t)$ for some continuous function $N : (t_0 - \delta, t_0 + \delta) \to \mathbb{Z}$ so that for $\delta > 0$ sufficiently small N is constant. Thus $\dot\beta(t_0) = \varphi\,(\partial/\partial\gamma)_{\beta(t_0)}$ hence β is nonspacelike (actually null). Also

$$F_{\theta,\beta(t_0)}\left(\left(T^\uparrow - S\right)_{\beta(t_0)}\,,\ \dot\beta(t_0)\right) = \frac{\varphi}{n+2} > 0$$

i.e. β is past directed and then $c' \leq c$. Together with $c << c''$ this implies (cf. [5], p. 22) $c' << c''$ and then $c'' \in I^+(c')$ thus yielding $I^+(c) \subset I^+(c')$. The roles of c, c' are interchangeable so the opposite inclusion holds too. Let $c'' \in I^-(c)$ so that $c'' << c$. Yet (by the proof of the statement on the chronological future map) $c \leq c'$ hence $c'' << c'$ i.e. $c'' \in I^-(c')$. Finally the causal future/past maps $J^\pm$ are constant on the fibres of π due to the transitivity of $\leq$.

iii) Since $\mathfrak{M}$ is not strongly causal its topology as a manifold contains strictly the Alexandrov topology. Q.e.d.

3.3. **Global differential geometry on $(\mathfrak{M}, F_\theta)$.** We shall need the following lemma (cf. [2]) relating the Levi-Civita connection D of $(\mathfrak{M}, F_\theta)$ to the Tanaka-Webster connection ∇ of (M, θ).

Lemma 1. *For any $X, Y \in H(M)$*

$$D_{X^\uparrow} Y^\uparrow = (\nabla_X Y)^\uparrow - (d\theta)(X,Y)T^\uparrow - (A(X,Y) + (d\sigma)(X^\uparrow, Y^\uparrow))S, \tag{4}$$

$$D_{X^\uparrow} T^\uparrow = (\tau X + \phi X)^\uparrow, \tag{5}$$

$$D_{T^\uparrow} X^\uparrow = (\nabla_T X + \phi X)^\uparrow + 2(d\sigma)(X^\uparrow, T^\uparrow)S, \tag{6}$$

$$D_{X^\uparrow} S = D_S X^\uparrow = (JX)^\uparrow, \tag{7}$$

$$D_{T^\uparrow} T^\uparrow = V^\uparrow, \quad D_S S = 0, \quad D_S T^\uparrow = D_{T^\uparrow} S = 0, \tag{8}$$

where $\phi : H(M) \to H(M)$ is given by $G_\theta(\phi X, Y) = (d\sigma)(X^\uparrow, Y^\uparrow)$, and $V \in H(M)$ is given by $G_\theta(V, Y) = 2(d\sigma)(T^\uparrow, Y^\uparrow)$.

Exterior differentiation of (2) leads to

$$(n+2)\, d\sigma = \pi^*\{i\, d\omega_\alpha{}^\alpha - \frac{i}{2}\, dg^{\alpha\overline{\beta}} \wedge dg_{\alpha\overline{\beta}} - \frac{1}{4(n+1)}\, d(\rho\theta)\}.$$

Using the identities $dg_{\alpha\overline{\beta}} = g_{\alpha\overline{\gamma}}\omega_{\overline{\beta}}{}^{\overline{\gamma}} + \omega_\alpha{}^\gamma g_{\gamma\overline{\beta}}$ (a consequence of $\nabla g_\theta = 0$) and $dg^{\alpha\overline{\beta}} = -g^{\gamma\overline{\beta}}g^{\alpha\overline{\rho}}dg_{\overline{\rho}\gamma}$ (a consequence of $g^{\alpha\overline{\beta}}g_{\overline{\beta}\gamma} = \delta^\alpha_\gamma$) one derives

$$dg^{\alpha\overline{\beta}} \wedge dg_{\alpha\overline{\beta}} = \omega_{\alpha\overline{\beta}} \wedge \omega^{\alpha\overline{\beta}} + \omega_{\overline{\alpha}\beta} \wedge \omega^{\overline{\alpha}\beta} = 0.$$

Also (cf. e.g. [19])

$$d\omega_\alpha{}^\alpha = R_{\lambda\overline{\mu}}\,\theta^\lambda \wedge \theta^{\overline{\mu}} + (W^\alpha_{\alpha\lambda}\,\theta^\lambda - W^\alpha_{\alpha\overline{\mu}}\,\theta^{\overline{\mu}}) \wedge \theta$$

where $R_{\lambda\overline{\mu}}$ is the pseudohermitian Ricci curvature and $W^\alpha_{\alpha\lambda}$ (respectively $W^\alpha_{\alpha\overline{\mu}}$) are certain contractions of covariant derivatives of $A^\alpha_{\overline{\beta}}$. Consequently

$$(n+2)G_\theta(\phi X, Y) = i(R_{\alpha\overline{\beta}}\,\theta^\alpha \wedge \theta^{\overline{\beta}})(X,Y) - \frac{\rho}{4(n+1)}\,(d\theta)(X,Y)$$

for any $X, Y \in H(M)$ or

$$\phi^{\overline{\alpha}\beta} = \frac{i}{2(n+2)}\left(R^{\overline{\alpha}\beta} - \frac{\rho}{2(n+1)}\,g^{\overline{\alpha}\beta}\right), \quad \phi^{\alpha\beta} = 0. \tag{9}$$

Similarly

$$(n+2)\,G_\theta(V,Y) = i\left(W^\alpha_{\alpha\overline{\mu}}\,\theta^{\overline{\mu}} - W^\alpha_{\alpha\lambda}\,\theta^\lambda\right)(Y) - \frac{1}{2(n+1)}\,Y(\rho)$$

for any $Y \in H(M)$ or

$$V^\alpha = \frac{1}{n+2}\left(iW^\gamma_{\gamma\overline{\beta}}g^{\overline{\beta}\alpha} - \frac{1}{2(n+1)}\,\rho^\alpha\right). \tag{10}$$

Let $M \subset \mathbb{C}^2$ be a nondegenerate real hypersurface and θ a contact form on M. Each chain of M is the projection via $\pi : C(M) \to M$ of some null geodesic of a metric in the restricted conformal class $[F_\theta]$ (cf. [11]). However not all null geodesics of $[F_\theta]$ project on chains. For example, a fibre of π is easily seen to be a null geodesic and its projection on M is a point. A *null chain* is the projection on M of a nonvertical null geodesic which is orthogonal to S. By a result of L.K. Koch every null geodesic of projects either to a point, or to a null chain, or to a chain of M (cf. Proposition 3.2 in [38], p. 250). If M is strictly pseudoconvex then all nonvertical null geodesics project to the chains of M.

4. Bundle completion of CR manifolds

4.1. **The Schmidt metric.** Let M be a strictly pseudoconvex CR manifold, of CR dimension n, and let θ be a contact form on M such that G_θ is positive definite. Let F_θ be the Fefferman metric on $\mathfrak{M} = C(M)$ and let D be the Levi-Civita connection of $(\mathfrak{M}, F_\theta)$. Let $\Pi_L : L(\mathfrak{M}) \to \mathfrak{M}$ be the principal $\mathrm{GL}(m,\mathbb{R})$-bundle of linear frames tangent to $\mathfrak{M}$ where $m = 2n+2$. A tangent vector $w \in T_u(L(\mathfrak{M}))$

is *D-horizontal* if there is a C^1 curve $\gamma : (-\delta, \delta) \to L(\mathfrak{M})$ such that $\gamma(0) = u$ and $\dot\gamma(0) = w$ and if $\gamma(t) = \big(\alpha(t), \{X_{j,\alpha(t)} : 1 \le j \le m\}\big)$ then $(D_{\dot\alpha} X_j)_{\alpha(t)} = 0$ for any $|t| < \delta$. Let $\Gamma_u \subset T_u(L(\mathfrak{M}))$ be the subspace of all D-horizontal tangent vectors. Then

$$T_u(L(\mathfrak{M})) = \Gamma_u \oplus \mathrm{Ker}(d_u \Pi_L), \quad (d_u R_a)\Gamma_u = \Gamma_{ua}\,, \tag{11}$$

for any $u \in L(\mathfrak{M})$ and $a \in \mathrm{GL}(m, \mathbb{R})$ i.e. Γ is a connection-distribution on $L(\mathfrak{M})$. For any left-invariant vector field $A \in \mathfrak{gl}(m, \mathbb{R})$ let $A^* \in \mathfrak{X}(L(\mathfrak{M}))$ be the corresponding fundamental vector field i.e. $A^*_u = (d_e L_u) A_e$ where $e = \left[\delta^i_j\right]_{1 \le i,j \le m} \in \mathrm{GL}(m, \mathbb{R})$. Also the map $L_u : \mathrm{GL}(m, \mathbb{R}) \to L(\mathfrak{M})$ is given by $L_u(a) = ua$ for any $a \in \mathrm{GL}(m, \mathbb{R})$. Let $\{E^i_j : 1 \le i, j \le m\}$ be the canonical basis of $\mathfrak{gl}(m, \mathbb{R}) \approx \mathbb{R}^{m^2}$. If $v_u : T_u(L(\mathfrak{M})) \to \mathrm{Ker}(d_u \Pi_L)$ is the projection associated to the decomposition (11) then let $\omega^i_j \in C^\infty(T^*(L(\mathfrak{M})))$ be the differential 1-forms determined by

$$\left(\omega^i_j\right)_u (X) \, \left(E^j_i\right)^*_u = v_u(X), \quad X \in T_u(L(\mathfrak{M})), \ \ u \in L(\mathfrak{M}).$$

Then $\omega = \omega^i_j \otimes E^j_i \in C^\infty(T^*(L(\mathfrak{M})) \otimes \mathfrak{gl}(m, \mathbb{R}))$ is the connection 1-form of $(\mathfrak{M}, F_\theta)$. Let $\Pi_O : O(\mathfrak{M}) \to \mathfrak{M}$ be the principal $\mathrm{O}(1, m-1)$-bundle of F_θ-orthonormal frames tangent to $\mathfrak{M}$, where $\mathrm{O}(1, m-1) \subset \mathrm{GL}(m, \mathbb{R})$ is the Lorentz group. Then $DF_\theta = 0$ implies that $\Gamma_u \subset T_u(O(\mathfrak{M}))$ for any $u \in O(\mathfrak{M})$ and $j^*\omega \in C^\infty(T^*(O(\mathfrak{M})) \otimes \mathfrak{o}(1, m-1))$ i.e. ω is actually $\mathfrak{o}(1, m-1)$-valued, where $j : O(\mathfrak{M}) \to L(\mathfrak{M})$ is the inclusion. In classical language (cf. e.g. [37], Vol. I, p. 83) ω is reducible to a connection form on $O(\mathfrak{M})$.

Let $B(e_i) \in \mathfrak{X}(O(\mathfrak{M}))$ be the standard horizontal lift associated to e_i where $\{e_i : 1 \le i \le m\}$ is the canonical linear basis in $\mathbb{R}^m$. That is $B(e_i)_u \in T_u(O(\mathfrak{M}))$ and $(d_u \Pi_O) B(e_i)_u = X_i$ for any $u = (c, \{X_j : 1 \le j \le m\}) \in O(\mathfrak{M})$ with $c = \Pi_O(u) \in \mathfrak{M}$. Then $\{B(e_i) : 1 \le i \le m\}$ is a (global) frame of Γ (thought of as a connection in $\mathrm{O}(1, m-1) \to O(\mathfrak{M}) \to \mathfrak{M}$). Next let $\{E_\alpha : 1 \le \alpha \le \ell\} \subset \mathfrak{o}(1, m-1)$ be an arbitrary linear basis in the Lie algebra of the Lorentz group ($\ell = m(m-1)/2$) so that $(E_\alpha)^*$ is a (global) frame of $\mathrm{Ker}(d\Pi_O)$. Given $u \in O(\mathfrak{M})$ and $X, Y \in T_u(O(\mathfrak{M}))$ we set

$$\gamma_u(X, Y) = \sum_{A=1}^{m+\ell} X^A Y^A \tag{12}$$

where $X = X^i B(e_i)_u + X^{m+\alpha} \left(E_\alpha\right)^*_u$ and $Y = Y^j B(e_j)_u + Y^{m+\beta} \left(E_\beta\right)^*_u$. We essentially follow the conventions in [34], p. 898 (itself based on the presentation in [29]). The original construction in [50] was to consider the canonical 1-form $\eta \in C^\infty(T^*(L(\mathfrak{M})) \otimes \mathbb{R}^m)$ given by $\eta_u = u^{-1} \circ (d_u \Pi_L)$ for any $u \in L(\mathfrak{M})$ and set

$$g_u(X,Y) = \omega_u(X) \cdot \omega_u(Y) + \eta_u(X) \cdot \eta_u(Y) \tag{13}$$

for any $X, Y \in T_u(L(\mathfrak{M}))$ and any $u \in L(\mathfrak{M})$. Cf. (3.1) in [50], p. 274 (or [14], p. 421). The dot products in (13) are respectively the Euclidean inner products in $\mathbb{R}^{m^2}$ and $\mathbb{R}^m$. The very definitions yield[6] $j^* g = \gamma$.

4.2. **The distance function.** As M is oriented so is $\mathfrak{M}$ hence $L(\mathfrak{M})$ has two connected components. Let $L^+(\mathfrak{M})$ be one of the connected components (an element $u \in L^+(\mathfrak{M})$ is a *positively oriented* linear frame) so that $\Pi_{L^+} = \Pi_L|_{L^+(\mathfrak{M})} : L^+(\mathfrak{M}) \to \mathfrak{M})$ is a $\mathrm{GL}^+(m)$-principal bundle [here $\mathrm{GL}^+(m)$ is the connected component of the identity in $\mathrm{GL}(m, \mathbb{R})$]. The approach in [50] was to consider the distance function $d_g : L^+(\mathfrak{M}) \to [0, +\infty)$ associated to the Riemannian metric g and take the Cauchy completion $\overline{L^+(\mathfrak{M})}$ of the (generally incomplete) metric space $(L^+(\mathfrak{M}), d_g)$. Then $\overline{L^+(\mathfrak{M})}$ is a complete metric space with the metric $\overline{d}_g$ given by

$$\overline{d}_g(u, v) = \lim_{\nu \to \infty} d_g(u_\nu \, , \, v_\nu)$$

where $\{u_\nu\}_{\nu \geq 1}$ and $\{v_\nu\}_{\nu \geq 1}$ are Cauchy sequences in $(L^+(\mathfrak{M}), d)$ representing $u, v \in \overline{L^+(\mathfrak{M})}$. Also the action of $\mathrm{GL}^+(m)$ on $L^+(\mathfrak{M})$ extends to an action of $\mathrm{GL}^+(m)$ as a topological group on $\overline{L^+(\mathfrak{M})}$. The quotient $\overline{L^+(\mathfrak{M})}/\mathrm{GL}^+(m)$ is then the *b-completion* of $\mathfrak{M}$ (cf. [50], p. 274-275). However the work in [24] shows that any G-structure on $\mathfrak{M}$ (in the sense of [13]) to which ω reduces, with $G \subset \mathrm{GL}(m, \mathbb{R})$ a closed subgroup, leads (by following essentially Schmidt's construction [50]) to the same completion (up to a homeomorphism). In particular let $\mathrm{O}^+(1, m-1)$ be the component of the identity in $\mathrm{O}(1, m-1)$ and $O^+(\mathfrak{M})$ a component of $O(\mathfrak{M})$ so that $\Pi_{O^+} : O^+(\mathfrak{M}) \to \mathfrak{M}$ is a principal $\mathrm{O}^+(1, m-1)$-bundle. Let $d_\gamma : O^+(\mathfrak{M}) \times O^+(\mathfrak{M}) \to [0, +\infty)$ be the distance function associated to the Riemannian metric γ and $\overline{O^+(\mathfrak{M})}$ the Cauchy completion of $(O^+(\mathfrak{M}), d_\gamma)$. Then $\overline{\mathfrak{M}} = \overline{O^+(\mathfrak{M})}/\mathrm{O}^+(1, m-1)$ is (homeomorphic to) the b-completion of $\mathfrak{M}$. For our purposes in this paper (as to building a b-completion and b-boundary for a CR manifold) we need

[6] This doesn't follow, as the reader should be aware, from a special choice of basis but rather from the fact that the identification with $\mathbb{R}^{m^2}$, and therefore the (first) dot product in (13), is relative to the chosen basis.

Lemma 2. *There is a natural free action of* $\mathrm{O}^+(1, m-1) \times S^1$ *on* $O^+(\mathfrak{M})$ *such that* $p = \pi \circ \Pi_{O^+} : O^+(\mathfrak{M}) \to M$ *is a principal bundle. Let* S *be the tangent to the* S^1*-action on* $\mathfrak{M}$*. Then*

$$\mathrm{Ker}(d_u p) = \mathrm{Ker}(d_u \Pi_{O^+}) \oplus \mathbb{R} S_u^{\uparrow}, \quad u \in O^+(\mathfrak{M}), \tag{14}$$

where $S^{\uparrow} \in \mathfrak{X}(O^+(\mathfrak{M}))$ *is the* Γ*-horizontal lift of* S*. Also if we set* $\beta_u = (d_u \Pi_{O^+} : \Gamma_u \to T_c(\mathfrak{M}))^{-1}$ *and*

$$\Gamma(\sigma)_u = \beta_u \, \mathrm{Ker}(\sigma_c), \quad u \in O^+(\mathfrak{M}), \quad c = \Pi_{O^+}(u) \in \mathfrak{M}, \tag{15}$$

then

$$T_u(O^+(\mathfrak{M})) = \Gamma(\sigma)_u \oplus \mathrm{Ker}(d_u p), \quad (d_u R_k)\Gamma(\sigma)_u = \Gamma(\sigma)_{R_k(u)}, \tag{16}$$

for any $u \in O^+(\mathfrak{M})$ *and* $k \in \mathrm{O}^+(1, m-1) \times S^1$ *i.e.* $\Gamma(\sigma)$ *is a connection in* $O^+(\mathfrak{M}) \xrightarrow{p} M$*. For each* $(a, \zeta) \in \mathrm{O}^+(1, m-1) \times S^1$ *the right translation* $R_{(a,\zeta)} : O^+(\mathfrak{M}) \to O^+(\mathfrak{M})$ *is uniformly continuous with respect to* d_γ*.*

Proof. We set $G = \mathrm{O}^+(1, m-1) \times S^1$ for simplicity. For each $\zeta \in S^1$ the right translation $R_\zeta : \mathfrak{M} \to \mathfrak{M}$ induces a diffeomorphism $\tilde{R}_\zeta : L(\mathfrak{M}) \to L(\mathfrak{M})$ given by

$$\tilde{R}_\zeta(u) = (R_\zeta(c), \, \{(d_c R_\zeta) X_j : 1 \le j \le m\})$$

for any linear frame $u = (c, \{X_j : 1 \le j \le m\}) \in L(\mathfrak{M})$. Continuity and $S^1 \subset \mathrm{Isom}(\mathfrak{M}, F_\theta)$ (cf. [41]) then imply $\tilde{R}_\zeta \left[O^+(\mathfrak{M}) \right] = O^+(\mathfrak{M})$. So $O^+(\mathfrak{M})$ admits a natural S^1-action. As it will be seen shortly the two actions commute i.e.

$$R_a \circ \tilde{R}_\zeta = \tilde{R}_\zeta \circ R_a, \quad a \in \mathrm{O}^+(1, m-1), \;\; \zeta \in S^1.$$

The product group G acts on $O^+(\mathfrak{M})$ by $u \cdot (a, \zeta) = R_a(\tilde{R}_\zeta(u))$. To check (14) note first that

$$\mathrm{Ker}(d_u \Pi_{O^+}) \cap \mathbb{R} S_u^{\uparrow} \subset \mathrm{Ker}(d_u \Pi_{O^+}) \cap \Gamma_u = (0)$$

hence the sum $\mathrm{Ker}(d_u \Pi_{O^+}) + \mathbb{R} S_u^{\uparrow}$ is direct. Moreover if $X \in \mathrm{Ker}(d_u p)$ then $(d_u \Pi_{O^+}) X \in \mathrm{Ker}(d_c \pi)$ i.e. $(d_u \Pi_{O^+}) X = \lambda S_c$ for some $\lambda \in \mathbb{R}$. Let us set $Y = X - \lambda S_u^{\uparrow} \in T_u(O^+(\mathfrak{M}))$. Then $(d_u \Pi_{O^+}) Y = 0$ hence $\mathrm{Ker}(d_u \Pi_{O^+}) \oplus \mathbb{R} S_u^{\uparrow} \subset \mathrm{Ker}(d_u p)$ and (14) follows by comparing dimensions.

Let $X \in \Gamma(\sigma)_u \cap \mathrm{Ker}(d_u p)$ so that $X = \beta_u Y$ for some $Y \in \mathrm{Ker}(\sigma_c)$ and $\mathrm{Ker}(d_c \pi) \ni (d_u \Pi_{O^+}) X = (d_u \Pi_{O^+}) \beta_u Y = Y$ hence $Y \in \mathrm{Ker}(\sigma_c) \cap \mathrm{Ker}(d_c \pi) = (0)$. So the sum $\Gamma(\sigma)_u + \mathrm{Ker}(d_u p)$ is direct and again a mere inspection of dimensions leads to the first formula in (16). To check the second formula let $k = (a, \zeta) \in G$. As Γ is $\mathrm{O}^+(1, m-1)$-invariant $(d_u R_a) \circ \beta_u = \beta_{ua}$. Also $\Pi_{O^+} \circ \tilde{R}_\zeta = R_\zeta \circ \Pi_{O^+}$, chain rule, and the S^1-invariance of Γ (as established later in this proof) yield $\beta_{\tilde{R}_\zeta(u)} \circ (d_c R_\zeta) = (d_u \tilde{R}_\zeta) \circ \beta_u$. Thus

$$(d_u R_k)\Gamma(\sigma)_u = (d_{ua}\tilde{R}_\zeta)(d_u R_a)\beta_u \mathrm{Ker}(\sigma_c) =$$
$$= (d_{ua}\tilde{R}_\zeta)\beta_{ua}\mathrm{Ker}(\sigma_c) = \beta_{\tilde{R}_\zeta(ua)}(d_c R_\zeta)\mathrm{Ker}(\sigma_c) =$$
$$= \beta_{\tilde{R}_\zeta(ua)}\mathrm{Ker}(\sigma_{R_\zeta(c)}) = \Gamma(\sigma)_{uk}\,.$$

If $c = [\omega] \in \mathfrak{M}$ for some $\omega \in \Lambda^{n+1,0}(M)_x \setminus \{0\}$ then $R_\zeta(c) = [\zeta\,\omega]$. Let $(u^j) \equiv (x^A\,,\,\gamma) : \pi^{-1}(U) \to \mathbb{R}$ be the local coordinate system on $\mathfrak{M}$ induced by (U, x^A) (a local coordinate system on M). Here $u^m = \gamma$ and the range of indices is $i, j, \cdots \in \{1, \cdots, m\}$ and $A, B, \cdots \in \{1, \cdots, 2n+1\}$. If $R^j = u^j \circ R_\zeta$ then

$$R^A(c) = u^A(c), \quad R^m(c) = \arg\left(\zeta\,\lambda/|\lambda|\right), \quad c = [\omega] \in \pi^{-1}(U).$$

Thus $R^m = \gamma + \arg(\zeta) + 2N\pi$ for some continuous function $N : \pi^{-1}(U) \to \mathbb{Z}$. Consequently $\partial R^j/\partial u^k = \delta^j_k$ on a sufficiently small neighborhood of each point $c \in \pi^{-1}(U)$. In particular $F_{ij} \circ R_\zeta = F_{ij}$ there so that $\Gamma^i_{jk} \circ R_\zeta = \Gamma^i_{jk}$ (here Γ^i_{jk} are the Christoffel symbols of F_θ). It follows that

$$\frac{\partial}{\partial u^j} - \left(\Gamma^i_{jk} \circ \Pi_L\right)\, X^k_\ell\, \frac{\partial}{\partial X^i_\ell}\,, \quad 1 \le j \le m, \tag{17}$$

are S^1-invariant (on a neighborhood of each point in $L(\mathfrak{M})$). Here $(u^j\,,\,X^i_j)$ are the naturally induced local coordinates on $L(\mathfrak{M})$ and (17) is a local frame of Γ defined on the open set $\Pi_L^{-1}(U)$. Therefore

$$(d_u\tilde{R}_\zeta)\Gamma_u = \Gamma_{\tilde{R}_\zeta(u)}\,, \quad u \in L(\mathfrak{M})\,, \quad \zeta \in S^1. \tag{18}$$

For each $\xi \in \mathbb{R}^m$ let $B(\xi) \in \mathfrak{X}(L(\mathfrak{M}))$ be the standard horizontal vector field associated to ξ. We shall need

$$\tilde{R}^*_\zeta \eta = \eta\,, \quad (d_u\tilde{R}_\zeta)A^*_u = A^*_{\tilde{R}_\zeta(u)} \tag{19}$$

for any $\zeta \in S^1$, $u \in L(\mathfrak{M})$ and $A \in \mathfrak{gl}(m, \mathbb{R})$ (cf. e.g. [13], p. 16). As a consequence of (18) the tangent vector $(d_u\tilde{R}_\zeta)B(\xi)_u - B(\xi)_{\tilde{R}_\zeta(u)}$ is horizontal. On the other hand (by the first relation in (19)) the same vector is also vertical hence

$$(d_u\tilde{R}_\zeta)B(\xi)_u = B(\xi)_{\tilde{R}_\zeta(u)}\,, \quad \xi \in \mathbb{R}^m. \tag{20}$$

Note that $\left\{\omega^i_j\,,\,\eta^j : 1 \le i, j \le m\right\}$ and $\left\{\left(E^i_j\right)^*\,,\,B(e_j) : 1 \le i, j \le m\right\}$ are dual. Hence (by (20) and the second relation in (19)) $\tilde{R}^*_z\omega^i_j = \omega^i_j$ and $\tilde{R}^*_\zeta\eta^j = \eta^j$. In particular $S^1 \subset \mathrm{Isom}(L(\mathfrak{M}), g)$ and then $S^1 \subset \mathrm{Isom}(O(\mathfrak{M}), \gamma)$. Combining this with a result of R.A. Johnson (cf.

Lemma 1.2 in [34], p. 898) it follows that for any compact subset $K \subset \mathrm{O}^+(1, m-1)$ there are constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\, d_\gamma(u, v) \leq d_\gamma(R_{(a,\zeta)}(u)\,,\, R_{(a,\zeta)}(v)) \leq \beta\, d_\gamma(u, v)$$

for any $u, v \in O^+(\mathfrak{M})$ and any $(a, \zeta) \in K \times S^1$. In particular $R_{(a,\zeta)}$ is uniformly continuous. Q.e.d.

4.3. **b-Completion, b-boundary.** We come now to the central notions in this paper. By Lemma 2 each right translation $R_{(a,\zeta)}$ extends uniquely to a uniformly continuous map $\overline{R}_{(a,\zeta)} : \overline{O^+(\mathfrak{M})} \to \overline{O^+(\mathfrak{M})}$ hence $G = \mathrm{O}^+(1, m-1) \times S^1$ acts as a topological group on $\overline{O^+(\mathfrak{M})}$. Let $\overline{M} = \overline{O^+(\mathfrak{M})}/G$ and $\overline{p} : \overline{O^+(\mathfrak{M})} \to \overline{M}$ be respectively the quotient space and projection. We endow $\overline{M}$ with the quotient topology i.e. the finest topology in which $\overline{p}$ is continuous. The injection $O^+(\mathfrak{M}) \hookrightarrow \overline{O^+(\mathfrak{M})}$ induces an injection $M = O^+(\mathfrak{M})/G \hookrightarrow \overline{M}$ and we set $\dot{M} = \overline{M} \setminus M$. Then $\overline{M}$ and $\dot{M}$ are referred to as the *b-completion* and *b-boundary* of M with respect to $(T_{1,0}(M), \theta)$. Also we say M is *b-complete* if $\dot{M} = \emptyset$ (otherwise M is *b-incomplete*). A few elementary topological properties of the b-completion $\overline{M}$ are listed in the following

Theorem 1. *Let M be a strictly pseudoconvex CR manifold and θ a contact form on M with G_θ positive definite. Let F_θ be the Fefferman metric on $\mathfrak{M} = C(M)$ and let d_γ be the corresponding distance function on $O^+(\mathfrak{M})$. Let $\overline{M} = \overline{O^+(\mathfrak{M})}/\,[\mathrm{O}^+(1, m-1) \times S^1]$ be the b-completion of (M, θ). Then*

i) *$G = \mathrm{O}^+(1, m-1) \times S^1$ acts transitively on the fibres of $\overline{p}$.*

ii) *$\overline{p}$ is an open map.*

iii) *Let $x \in M$ and let us endow the fibre $p^{-1}(x)$ with the metric induced by the b-metric γ. Then $p^{-1}(x)$ is complete.*

iv) *Let us set $\rho(x, y) = \inf\{\overline{d_\gamma}(u, v) : u \in \overline{p}^{-1}(x),\quad v \in \overline{p}^{-1}(y)\}$. Then ρ is a semi-metric on $\overline{M}$ and the ρ-topology is contained in the quotient topology.*

v) *If $(O^+(\mathfrak{M}), d_\gamma)$ is a complete metric space and if ρ is metric then (M, ρ) is complete.*

vi) *If the orbits of G are not closed in $\overline{O^+(\mathfrak{M})}$ then $\overline{M}$ is not T_1.*

vii) *$\overline{M}$ is Hausdorff if and only if* $\mathrm{graph}\,(G)$ *is closed in $\overline{O^+(\mathfrak{M})} \times \overline{O^+(\mathfrak{M})}$.*

viii) *Let $\{u_\nu\}_{\nu \geq 1} \subset O^+(\mathfrak{M})$ be a Cauchy sequence without limit in $O^+(\mathfrak{M})$. Let us assume that there is a compact subset $K \subset M$ such that $\{p(u_\nu)\}_{\nu \geq 1} \subset K$. Then $p^{-1}(x_0)$ is incomplete with respect to d_γ for some $x_0 \in M$. Consequently $\overline{M}$ is at most T_0.*

Proof. The proofs are imitative of [20] and [50]. (i) is immediate.

(ii) Let $U \subset \overline{M}$ be an open set. For each $k \in G$ the right translation $\overline{R}_k : \overline{O^+(\mathfrak{M})} \to \overline{O^+(\mathfrak{M})}$ is a homeomorphism so $U = \overline{R}_{k^{-1}} \overline{R}_k(U)$ is open and hence $\overline{R}_k(U)$ is open. Transitivity of G on the fibres of $\overline{p}$ implies $\overline{p}^{\,-1}(\overline{p}(U)) = \bigcup_{k \in G} \overline{R}_k(U)$. Hence $\overline{p}(U)$ is open in $\overline{M}$.

(iii) The fibre $p^{-1}(x)$ carries the metric $\gamma_x = \iota^* \gamma$ where $\iota : p^{-1}(x) \to O^+(\mathfrak{M})$ is the inclusion. Therefore the associated distance function d_{γ_x} is computed by taking the greatest lower bound over lengths of piecewise C^1 curves contained in $p^{-1}(x)$ (rather than restricting d_γ to $p^{-1}(x) \times p^{-1}(x)$). Let $\{u_\nu\}_{\nu \geq 1}$ be a Cauchy sequence in $(p^{-1}(x), d_{\gamma_x})$. For any $\epsilon > 0$ there is $\nu_\epsilon \geq 1$ such that

$$d_\gamma(u_\nu\,,\,u_\mu) \leq d_{\gamma_x}(u_\nu\,,\,u_\mu) < \epsilon, \quad \nu, \mu \geq \nu_\epsilon\,,$$

i.e. $\{u_\nu\}_{\nu \geq 1}$ is Cauchy in $(O^+(\mathfrak{M}), d_\gamma)$ as well. Let $u = \lim_{\nu \to \infty} u_\nu \in \overline{O^+(\mathfrak{M})}$ be the corresponding class of equivalence and let us set $y = \overline{p}(u) \in \overline{M}$. If $y = x$ then $u \in \overline{p}^{\,-1}(x) = p^{-1}(x)$ (fibres of p and $\overline{p}$ over points of M coincide). The possibility $y \neq x$ may be ruled out as follows. If that is the case then $p^{-1}(x)$ and $\overline{p}^{\,-1}(y)$ are disjoint sets hence $r = \overline{d}_\gamma(u, p^{-1}(x)) > 0$. Then the open ball $B_{\overline{d}_\gamma}(u, r/2) = \{v \in \overline{O^+(\mathfrak{M})} : \overline{d}_\gamma(u, v) < r/2\}$ contains u yet doesn't meet $\overline{p}^{\,-1}(x)$. Hence $\{u_\nu\}_{\nu \geq 1}$ doesn't meet $B_{\overline{d}_\gamma}(u, r/2)$ in contradiction with the convergence of $\{u_\nu\}_{\nu \geq 1}$ to u.

(iv) For notions and results in general topology we rely on [61]. Clearly ρ is a semi-metric on $\overline{M}$. It determines a topology on $\overline{M}$ for which the cells $N(x, \epsilon) = \{y \in \overline{M} : \rho(x, y) < \epsilon\}$ form a basis of open sets. The quotient topology is the family of sets

$$\{U \subset \overline{M} : \overline{p}^{\,-1}(U) \text{ is open in } \overline{O^+(\mathfrak{M})}\}.$$

The quotient topology contains the ρ-topology because $\overline{p}$ is continuous in the ρ-topology. Indeed let $u_0 \in \overline{O^+(\mathfrak{M})}$ and $x_0 = \overline{p}(u_0) \in \overline{M}$. Given an arbitrary cell $N(x_0, \epsilon)$ let $0 < r < \epsilon$. Then for any $u \in N(u_0, r)$

$$\epsilon > r > \overline{d}_\gamma(u, u_0) \geq \overline{d}_\gamma(\overline{p}^{\,-1}(x), \overline{p}^{\,-1}(x_0)) = \rho(x, x_0)$$

where $x = \overline{p}(c)$.

(v) Here M is thought of as a metric space carrying the distance function induced by ρ. Let $\{x_\mu\}_{\mu \geq 1}$ be a Cauchy sequence in (M, ρ). There is a subsequence $\{y_\nu\}_{\nu \geq 1}$ of $\{x_\mu\}_{\mu \geq 1}$ such that

$$\rho(y_\nu\,,\,y_{\nu+1}) < \left(\frac{1}{2}\right)^\nu, \quad \nu \geq 1.$$

That is $d_\gamma(p^{-1}(y_\nu), p^{-1}(y_{\nu+1})) < (1/2)^\nu$ hence for each $\nu \geq 1$ one may choose $u_\nu \in p^{-1}(y_\nu)$ such that $d_\gamma(u_\nu\,,\,u_{\nu+1}) < (1/2)^\nu$. Consequently

$$d_\gamma(u_\nu\,,\,u_{\nu+\mu}) \leq \sum_{i=1}^{\mu} \left(\frac{1}{2}\right)^{\nu+i-1} < \left(\frac{1}{2}\right)^{\nu-1}$$

so that $\{u_\nu\}_{\nu\geq 1}$ is Cauchy in $(O^+(\mathfrak{M}), d)$. By assumption the space $(O^+(\mathfrak{M}), d_\gamma)$ is complete so there is a unique $u_\infty \in O^+(\mathfrak{M})$ such that $\lim_{\nu\to\infty} u_\nu = u_\infty$. Let $y_\infty = p(u_\infty) \in M$. Finally

$$\rho(y_\nu, y_\infty) = d_\gamma(p^{-1}(y_\nu), p^{-1}(y_\infty)) \leq d_\gamma(u_\nu, u_\infty) \to 0, \quad \nu \to \infty,$$

so that $\{x_\mu\}_{\mu\geq 1}$ is convergent (as a Cauchy sequence containing a convergent subsequence).

(vi) Let us assume that $\overline{M}$ is T_1 hence the singleton $\{x\} \subset \overline{M}$ is closed for any $x \in \overline{M}$. Thus $\overline{M} \setminus \{x\}$ is open so that $\overline{O^+(\mathfrak{M})} \setminus \overline{p}^{\,-1}(x) = \overline{p}^{\,-1}\left(\overline{M} \setminus \{x\}\right)$ is open i.e. $\overline{p}^{\,-1}(x)$ is closed in $\overline{O^+(\mathfrak{M})}$.

(vii) Let $\mathcal{G} = \operatorname{graph}(G)$ be the graph of the G-action on $\overline{O^+(\mathfrak{M})}$ i.e. $\mathcal{G} = \{(u, \overline{R}_k(u)) : u \in \overline{O^+(\mathfrak{M})},\quad k \in G\}$. We assume that $\overline{M}$ is Hausdorff and consider a point $(u, v) \in \overline{O^+(\mathfrak{M})} \times \overline{O^+(\mathfrak{M})} \setminus \mathcal{G}$. Let $x = \overline{p}(u)$ and $y = \overline{p}(v)$ so that $x \neq y$ (if $x = y$ then u and v are equivalent mod G i.e. (x, y) lies on $\mathcal{G}$). As $\overline{M}$ is T_2 there exist open sets $U \subset \overline{M}$ and $V \subset \overline{M}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Thus $\overline{p}^{\,-1}(U) \times \overline{p}^{\,-1}(V)$ is an open neighborhood of (u, v) contained in $\overline{O^+(\mathfrak{M})} \times \overline{O^+(\mathfrak{M})} \setminus \mathcal{G}$ i.e. (u, v) is an interior point. To prove sufficiency we assume that $\mathcal{G}$ is closed and consider $x, y \in \overline{M}$ with $x \neq y$. Let $u \in \overline{p}^{\,-1}(x)$ and $v \in \overline{p}^{\,-1}(y)$ so that u, v are not equivalent $\operatorname{mod} G$. Thus $(u, v) \notin \mathcal{G}$ i.e. there exist open sets $\mathcal{U}$ and $\mathcal{V}$ in $\overline{O^+(\mathfrak{M})}$ such that $(u, v) \in \mathcal{U} \times \mathcal{V} \subset \overline{O^+(\mathfrak{M})} \times \overline{O^+(\mathfrak{M})} \setminus \mathcal{G}$. The projection $\overline{p} : \overline{O^+(\mathfrak{M})} \to \overline{M}$ is an open map (cf. (ii) above) hence $U = \overline{p}(\mathcal{U})$ and $V = \overline{p}(\mathcal{V})$ are open sets in $\overline{M}$. Also $U \cap V = \emptyset$ because the G-action is transitive on the fibres of $\overline{p}$.

(viii) As K is a compact set there is a subsequence $\{v_\nu\}_{\nu\geq 1}$ of $\{u_\nu\}_{\nu\geq 1}$ such that $\{p(v_\nu)\}_{\nu\geq 1}$ converges to some $x \in M$. We shall show that $p^{-1}(x)$ is incomplete with respect to (the restriction to $p^{-1}(x) \times p^{-1}(x)$ of) d_γ. The proof is by contradiction. If $C = p^{-1}(x)$ is complete then C is a closed subset of $O^+(\mathfrak{M})$. Also $\{u_\nu\}_{\nu\geq 1}$ is not contained in C (if it were it would have a limit there). Thus there is $\nu_0 \geq 1$ such that $d_\gamma(v_\nu\,,\,C) > 0$ for any $\nu \geq \nu_0$. Let $f_\nu : O^+(\mathfrak{M}) \to \mathbb{R}$ be defined by $f_\nu(w) = d_\gamma(v_\nu\,,\,w)$ for any $w \in O^+(\mathfrak{M})$. Then f_ν is continuous and C closed so that $\inf_{w\in C} f_\nu(w)$ is realized in C i.e. for each $\nu \geq \nu_0$ there is $w_\nu \in C$ such that $d_\gamma(v_\nu\,,\,w_\nu) = d_\gamma(v_\nu\,,\,C)$. Note that $p(v_\nu) \to x$ implies

$d_\gamma(v_\nu\,,\,C) \to 0$ as $\nu \to \infty$. Then on one hand $\{w_\nu\}_{\nu\geq 1}$ is Cauchy for

$$d_\gamma(w_\nu\,,\,w_\mu) \leq d_\gamma(w_\nu\,,\,v_\nu) + d_\gamma(v_\nu\,,\,v_\mu) + d_\gamma(v_\mu\,,\,w_\mu) < \epsilon$$

for any $\nu, \mu \geq \nu_\epsilon$, and on the other hand $d_\gamma(v_\nu\,,\,w_\nu) \to 0$ means that $\{v_\nu\}_{\nu\geq 1}$ and $\{w_\nu\}_{\nu\geq 1}$ are equivalent Cauchy sequences so they must represent the same point $\lim_{\nu\to\infty} w_n = \lim_{\nu\to\infty} v_\nu \in \overline{O^+(\mathfrak{M})} \setminus O^+(\mathfrak{M})$ which implies that C is not complete. To prove the second statement in (viii) we shall pin down, under the given assumptions, two elements $x, y \in \overline{M}$ with $x \neq y$ such that all open neighborhoods of y contain x as well. By the first statement in (viii) there is $x \in M$ such that $p^{-1}(x)$ is incomplete with respect to the full metric d. Therefore $p^{-1}(x)$ contains at least a Cauchy sequence $\{w_\nu\}_{\nu\geq 1}$ without limit there. Let $w = \lim_{\nu\to\infty} w_\nu \in \overline{O^+(\mathfrak{M})} \setminus O^+(\mathfrak{M})$. Then w lies on the topological boundary of $p^{-1}(x)$ (as a subset of $\overline{O^+(\mathfrak{M})}$) hence any open neighborhood U of w intersects $p^{-1}(x)$. Let $y = \overline{p}(w) \in \overline{M}$. Finally if $V \subset \overline{M}$ is an arbitrary open neighborhood of y then $\overline{p}^{\,-1}(V)$ is an open neighborhood of w hence $\overline{p}^{\,-1}(V) \cap p^{-1}(x) \neq \emptyset$ hence $x \in V$. Q.e.d.

A comment is in order on the perhaps a bit subtle difference between statements (iii) and (viii) in Theorem 1: γ_x is the first fundamental form of $p^{-1}(x)$ as a submanifold of $(O^+(\mathfrak{M}), \gamma)$. The distance function d_{γ_x} (associated to the Riemannian metric γ_x) and the restriction of d_γ to $p^{-1}(x) \times p^{-1}(x)$ do not coincide[7] in general (completeness in (iii) and (viii) is relative to distinct distance functions).

An almost *verbatim* repetition of the arguments in the proofs of Lemma 2 and Theorem 1 leads to

Corollary 1. *The S^1 action on $\mathfrak{M}$ extends to a unique uniformly continuous topological S^1 action on $\overline{\mathfrak{M}}$ leaving $\dot{\mathfrak{M}}$ invariant. Then $\overline{M} = \overline{\mathfrak{M}}/S^1$ and $\dot{M} = \dot{\mathfrak{M}}/S^1$. Let $\overline{\pi} : \overline{\mathfrak{M}} \to \overline{M}$ and $\dot{\pi} : \dot{\mathfrak{M}} \to \dot{M}$ be the canonical projections. Then the fibres of $\overline{\pi}$ over b-boundary points are contained in the b-boundary $\dot{\mathfrak{M}}$ and S^1 acts transitively on the fibres of $\dot{\pi}$. The projection $\overline{\pi}$ is an open map. If the orbits of S^1 are not closed in $\overline{\mathfrak{M}}$ then $\overline{M}$ is not T_1. $\overline{M}$ is Hausdorff if and only if* graph(S^1) *is closed in $\overline{\mathfrak{M}} \times \overline{\mathfrak{M}}$.*

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[7]This is of course a general fact in the theory of isometric immersions among Riemannian manifolds. If $j : \mathfrak{S} \to \mathfrak{N}$ is an immersion of a manifold $\mathfrak{S}$ into a Riemannian manifold $(\mathfrak{N}, \gamma)$ then in general one has but $d_\gamma(x, y) \leq d_{j^*\gamma}(x, y)$ for any $x, y \in \mathfrak{S}$ (d_γ measures distances among $x, y \in \mathfrak{S}$ by measuring lengths of arbitrary piecewise C^1 curves joining x and y while $d_{j^*\gamma}$ is "confined" to curves lying in $\mathfrak{S}$).

Example 3. (*Heisenberg group*) Let $\mathbb{H}_1 = \mathbb{C} \times \mathbb{R}$ be the Heisenberg group (cf. e.g. [19], p. 11-12) carrying the CR structure spanned by $L = \partial/\partial z + i\overline{z}\,\partial/\partial t$ ($\overline{L}$ is the unsolvable *Lewy operator*) and the contact form $\theta = dt + i(z\,d\overline{z} - \overline{z}\,dz)$. The relationship to Example 1 is well known: the map $f : \mathbb{H}_1 \to \partial\Omega_0$, $f(z,t) = (z\,,\,t + i|z|^2)$, $(z,t) \in \mathbb{H}_1$, is a CR isomorphism i.e. a C^∞ diffeomorphism such that $f_* L = \partial/\partial z - 2\overline{z}F_0\partial/\partial w$. The Fefferman metric is then given by

$$F_\theta = 2\left\{dx^2 + dy^2\right\} + \frac{2}{3}\left\{dt + 2(x\,dy - y\,dx)\right\} \odot d\gamma$$

where $z = x + iy$ and $C(\mathbb{H}_1)$ is a 4-dimensional space-time with the time orientation $(\partial/\partial t)^\uparrow - (3/2)\,\partial/\partial\gamma$. The horizontal lift is taken with respect to the connection 1-form $(1/3)\,d\gamma$. The b-completion

$$\overline{\mathbb{H}}_1 = \overline{O^+(C(\mathbb{H}_1))}/[\mathrm{O}^+(1,3) \times S^1] = \overline{C(\mathbb{H}_1)}/S^1$$

and b-boundary $\dot{\mathbb{H}}_1 = \overline{\mathbb{H}}_1 \setminus \mathbb{H}_1$ are then well defined. The analysis of Example 3 is completed in § 5 where we show that $\dot{\mathbb{H}}_1 = \emptyset$. $\square$

Example 4. (*Penrose's twistor CR manifold*) Let $\mathbb{T} = (\mathbb{C}^4, \Sigma)$ be the twistor space (cf. [46]) i.e.

$$\Sigma(W) = \overline{W}\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} W^t\,, \quad W \in \mathbb{C}^4,$$

(I_2 is the unit 2×2 matrix). Let $\mathbb{T}_0 = \{W \in \mathbb{T} : \Sigma(W) = 0\}$ so that $\mathbb{T}_0$ is a CR manifold of CR dimension $n = 3$. The signature of the Levi form $\partial\overline{\partial}\Sigma$ of $\mathbb{T}_0$ is $(+\,-\,0)$ hence each pseudohermitian structure $\theta \in C^\infty(H(\mathbb{T}_0)^\perp)$ is degenerate[8]. The methods of pseudohermitian geometry (cf. [57]) may however be applied to $\mathbb{P}(\mathbb{T}_0)$ as will be seen shortly. The projective twistor space is $\mathbb{P}(\mathbb{T}) = (\mathbb{T} \setminus \{0\})\,/(\mathbb{C}\setminus\{0\})$. We set $\mathbb{P}(\mathbb{T}_0) = \{[W] \in \mathbb{P}(\mathbb{T}) : W \in \mathbb{T}_0\setminus\{0\}\}$. Then $\mathbb{P}(\mathbb{T}_0)$ is a CR manifold (with the CR structure induced by the complex structure on $\mathbb{P}(\mathbb{T})$) and the projection $\mathbb{T}\setminus\{0\} \to \mathbb{P}(\mathbb{T})$ descends to a CR map $\mathbb{T}_0\setminus\{0\} \to \mathbb{P}(\mathbb{T}_0)$. Let $I = \{[W] \in \mathbb{P}(\mathbb{T}) : W_0 = W_1\}$ (a projective line). Let $\Lambda = \{v \in \mathbb{R}^4 \setminus \{0\} : -v_0^2 + \sum_{i=1}^2 v_i^2 = 0\}$ be the null cone. Let $\mathbb{M} = (\mathbb{R}^4, \eta)$ be the Minkowski space i.e. $\eta(x,y) = -x_0y_0 + \sum_{i=1}^3 x_iy_i$ for any $x, y \in \mathbb{R}^4$. Given $x \in \mathbb{M}$ and $v \in \Lambda$ let $N_{x,v} = \{x + tv : t \in \mathbb{R}\}$ be the null geodesic in $\mathbb{M}$ of initial data (x,v). Also let $\Omega_0 = \{N_{x,v} : x \in \mathbb{M},\ v \in \Lambda\}$ (a bundle of null cones over $\mathbb{M}$). The fibre $(\Omega_0)_x$, i.e. the set of null geodesics (light rays) through x, is the field of vision of an observer situated at x. There is a natural identification $\mathbb{P}(\mathbb{T}_0) \setminus I \approx \Omega_0$ (cf. [46] or [19], p. 24).

[8] I.e. the Levi form L_θ has a nontrivial null space.

Let $S^1 \to C(\mathbb{P}(\mathbb{T}_0)) \to \mathbb{P}(\mathbb{T}_0)$ be the canonical circle bundle over $\mathbb{P}(\mathbb{T}_0)$. Let us set $\rho(\zeta) = \zeta_2 + \overline{\zeta}_2 + \zeta_1\overline{\zeta}_3 + \overline{\zeta}_1\zeta_3$. Let $U_0 = \{[W] \in \mathbb{P}(\mathbb{T}) : W_0 \neq 0\}$ with the canonical complex coordinates $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ so that $\mathbb{P}(\mathbb{T}_0) \cap U_0$ is described by $\rho(\zeta) = 0$. Thus $T_{1,0}(\mathbb{P}(\mathbb{T}_0))$ is locally the span of $T_1 = \partial/\partial\zeta_1 - \overline{\zeta}_3\,\partial/\partial\zeta_2$ and $T_2 = \partial/\partial\zeta_3 - \overline{\zeta}_1\,\partial/\partial\zeta_2$. Then the local components of the Levi form are $g_{1\overline{1}} = g_{2\overline{2}} = 0$ and $g_{1\overline{2}} = g_{2\overline{1}} = 1/2$ (so that $\mathbb{P}(\mathbb{T}_0)$ is nondegenerate of signature $(+-)$) hence the Fefferman metric $F_\theta = \pi^*\tilde{G}_\theta + \frac{1}{2}\,(\pi^*\theta) \odot d\gamma$ is a semi-Riemannian metric of index 3 on $C(\mathbb{P}(\mathbb{T}_0))$ (here $\theta = \frac{i}{2}(\overline{\partial} - \partial)\rho$). The constructions in § 3 generalize easily to the case where the Fefferman metric is a semi-Riemannian metric of arbitrary index $2s+1$. Actually the constructions depend only on D (or more generally only on the full parallelism structure on $O^+(\mathfrak{M})$ associated to $\Gamma(\sigma)$, cf. also [16]). If $M = \mathbb{P}(\mathbb{T}_0) \setminus I$ (an open subset of $\mathbb{P}(\mathbb{T}_0)$) then its b-completion and b-boundary are

$$\overline{M} = \overline{O^+(C(M))}/[\mathrm{O}^+(3,3) \times S^1] = \overline{C(M)}/S^1\,, \quad \dot{M} = \overline{M} \setminus M.$$

The physical meaning of points in $\dot{M}$ is unknown. The b-boundary (in the sense of [50]) of the Minkowski space is known to be empty ($\dot{\mathbb{M}} = \emptyset$). However as shown in [45], p. 337, each C^∞ function with values in the Argand plane $z : \mathbb{M} \to \mathbb{C} \cup \{\infty\}$ gives rise to a C^∞ section $s : \mathbb{M} \to \Omega_0$ so that $\mathbb{M}$ embeds in M. Then $\mathbb{M}$ carries the physical field s^*g_θ and singular points should arise. The Reeb vector of $(M \cap U_0, \theta)$ is $T = i\left(\partial/\partial\zeta_2 - \partial/\partial\overline{\zeta}_2\right)$. Also $\{T_1 + \overline{T}_1\,,\, T_2 + \overline{T}_2\,,\, i\left(T_1 - \overline{T}_1\right),\, i\left(T_2 - \overline{T}_2\right)\}$ is a local frame of $H(M)$ with respect to which the Webster metric is

$$g_\theta : \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The problem of relating the b-boundary of $\mathbb{M}_s = (\mathbb{R}^4,\, s^*g_\theta)$ to $\dot{M}$ is open. Picking up a section $s : \mathbb{M} \to \Omega_0$ amounts to choosing smoothly one light ray in the field of vision $(\Omega_0)_x$ over each $x \in \mathbb{M}$. One may then start with a C^∞ function $v : \mathbb{M} \to \Lambda$ and set $s(x) = \{x + t\,v(x) : t \in \mathbb{R}\}$. Of course the question arises whether a *canonical* choice of such v is feasible. For instance, as $v : \mathbb{M} \to \Lambda$ may be thought of as a vector field (of null vectors) tangent to $\mathbb{M}$, one may request that v be an extremal of some classical action (such as the total bending or biegung, cf. e.g. [60], in one of its known Lorentzian generalizations, cf. e.g. [25] or [18]). Cf. also [6]. Let us consider a C^∞ function $\lambda : \mathbb{M} \to \mathbb{R} \setminus \{1\}$ and

$s : \mathbb{M} \to \mathbb{P}(\mathbb{T}_0) \setminus \{I\}$ given by

$$s(x) = \left[(1, \lambda(x)) \, , \frac{1}{i\sqrt{2}} \, (1, \lambda(x)) \, \phi(x) \right] , \quad x \in \mathbb{M}. \tag{21}$$

Then $\Sigma(s(x)) = 0$ and $s(x) \in I \iff \lambda(x) = 1$ i.e. $s(x)$ is well defined. Note that $z = 1 + i\lambda : \mathbb{M} \to \mathbb{C}$ is the function required by the approach in [45]. The study of the geometry of the second fundamental form of (21) is open. A knowledge of that is likely to lead to an useful relationship among $\overline{\mathbb{M}}_s$ and $\overline{\mathbb{P}(\mathbb{T}_0) \setminus I}$. □

5. b-Incomplete curves

5.1. **Bundle length, b-incomplete curves.** Let $I \subset \mathbb{R}$ be a bounded interval such that $0 \in I$. According to [50] (slightly reformulated with O^+ replacing L^+) a curve $c : I \to \mathfrak{M}$ is said to have *finite bundle length* if there is $u \in O^+(\mathfrak{M})$ such that the Γ-horizontal lift $c^* : I \to O^+(\mathfrak{M})$ issuing at u has finite length with respect to the b-metric γ i.e.

$$L(c^*) = \int_I \left\| c^*(t)^{-1} \dot{c}(t) \right\| \, dt < \infty.$$

Here $\|\xi\|$ is the Euclidean norm of $\xi \in \mathbb{R}^m$. Cf. also [14], p. 437. A curve $c : [0, 1) \to \mathfrak{M}$ is *b-incomplete* if it has finite bundle length and admits no continuous extension to a map $[0, 1] \to \mathfrak{M}$ (i.e. c is inextensible). The relevance of this class of curves consists in the fact that the b-boundary $\dot{\mathfrak{M}} = \overline{\mathfrak{M}} \setminus \mathfrak{M}$ consists precisely of the end points in $\overline{\mathfrak{M}}$ of b-incomplete curves in $\mathfrak{M}$.

Similar considerations as to the b-boundary $\dot{M}$ require the connection $\Gamma(\sigma)$ in $G \to O^+(\mathfrak{M}) \xrightarrow{p} M$ (cf. Lemma 2). We adopt the following definition. A curve $\alpha : I \to M$ has *finite b-length* if there is $u \in O^+(\mathfrak{M})$ such that the $\Gamma(\sigma)$-horizontal lift $\alpha^\uparrow : I \to O^+(\mathfrak{M})$ issuing at $\alpha^\uparrow(0) = u$ has finite length with respect to the b-metric γ. Also $\alpha : [0, 1) \to M$ is *b-incomplete* if it has finite bundle length and admits no continuous extension to a map $[0, 1] \to M$.

Theorem 2. *For any b-incomplete curve $\alpha : [0, 1) \to M$ its end point $\lim_{t \to 1^-} \alpha(t)$ exists in $\overline{M}$ and lies on the b-boundary $\dot{M}$. Conversely, any point on $\dot{M}$ is an endpoint of some b-incomplete curve.*

Proof. As α has finite bundle length there is $u \in O^+(\mathfrak{M})$ such that the horizontal lift $\alpha^\uparrow : [0, 1) \to O^+(\mathfrak{M})$ with $\alpha^\uparrow(0) = u$ has finite length with respect to γ. Let $c = \Pi_{O^+} \circ \alpha^\uparrow : [0, 1) \to \mathfrak{M}$. As

$$\dot{\alpha}^\uparrow(t) \in \Gamma(\sigma)_{\alpha^\uparrow(t)} = \beta_{\alpha^\uparrow(t)} \mathrm{Ker}(\sigma_{c(t)}) \subset \Gamma_{\alpha^\uparrow(t)}, \quad 0 \le t < 1, \tag{22}$$

one has

$$\gamma_{\alpha^{\uparrow}(t)}(\dot{\alpha}^{\uparrow}(t), \dot{\alpha}^{\uparrow}(t)) = \|\eta_{\alpha^{\uparrow}(t)}\left(\dot{\alpha}^{\uparrow}(t)\right)\|^2 = \|\alpha^{\uparrow}(t)^{-1}\dot{c}(t)\|^2$$

and $L(\alpha^{\uparrow}) < \infty$ yields

$$\lim_{\nu\to\infty} \int_0^{t_\nu} \left\|\alpha^{\uparrow}(t)^{-1}\dot{c}(t)\right\| \, dt = L_0$$

for some $L_0 \in \mathbb{R}$ and any sequence $\{t_\nu\}_{\nu\geq 1} \subset [0,1)$ such that $\lim_{\nu\to\infty} t_\nu = 1$. In particular for any $\epsilon > 0$ there is $\nu_\epsilon \geq 1$ such that

$$d_\gamma(\alpha^{\uparrow}(t_\nu)\, , \, \alpha^{\uparrow}(t_{\nu+\mu})) \leq \left| \int_{t_\nu}^{t_{\nu+\mu}} \left\|\alpha^{\uparrow}(t)^{-1}\dot{c}(t)\right\| \, dt \right| < \epsilon$$

for any $\nu \geq \nu_\epsilon$ and any $\mu \geq 1$ i.e. $\{\alpha^{\uparrow}(t_\nu)\}_{\nu\geq 1}$ is a Cauchy sequence in $(O^+(\mathfrak{M}), d_\gamma)$. Let $u_0 \in \overline{O^+(\mathfrak{M})}$ be its equivalence class, so that $\lim_{\nu\to\infty} \alpha^{\uparrow}(t_\nu) = u_0$ in $\overline{O^+(\mathfrak{M})}$. Let $x_0 = \overline{p}(u_0) \in \overline{M}$. By the continuity of $\overline{p}$ one has $\lim_{\nu\to\infty} \alpha(t_\nu) = x_0$ in $\overline{M}$. As α is b-incomplete it must be that $x_0 \in \overline{M} \setminus M$. Viceversa, let $\overline{x} \in \dot{M}$ and let $\overline{c} \in \dot{\mathfrak{M}}$ such that $\overline{\pi}(\overline{c}) = \overline{x}$. By a result in [50] there is a b-incomplete curve $\rho : [0,1) \to \mathfrak{M}$ such that $\lim_{t\to 1^{-1}} \rho(t) = \overline{c}$ i.e. its Γ-horizontal lift $\rho^* : [0,1) \to O^+(\mathfrak{M})$ issuing at some $u \in \Pi^{-1}_{O^+}(\gamma(0))$ has finite length with respect to γ. On the other hand if $\alpha = \pi \circ \rho$ and $\alpha^{\uparrow} : [0,1) \to O^+(\mathfrak{M})$ is the $\Gamma(\sigma)$-horizontal lift of α issuing at u then (again by (22) i.e. $\alpha^{\uparrow}$ is $\Gamma(\sigma)$-horizontal, and hence Γ-horizontal) $\alpha^{\uparrow} = \rho^*$ (as two Γ-horizontal curves issuing at the same point). Q.e.d.

5.2. **Inextensible Fefferman space-times.** A space-time $(\mathfrak{N}, F)$ is an *extension* of the Fefferman space-time if $\mathfrak{M}$ is an open set in $\mathfrak{N}$ and $\iota^* F = F_\theta$ where $\iota : \mathfrak{M} \to \mathfrak{N}$ is the inclusion. Also $(\mathfrak{M}, F_\theta)$ is *inextensible* if it has no extension. A strictly pseudoconvex pseudohermitian manifold $(M, T_{1,0}(M), \theta)$ is *Fefferman inextensible* if the space-time $(\mathfrak{M}, F_\theta, T^{\uparrow} - S)$ is inextensible. A CR manifold $(N, T_{1,0}(N))$ is an *extension* of $(M, T_{1,0}(M))$ if $M \subset N$ is an open subset and the inclusion $\iota : M \to N$ is a CR map. Then $(M, T_{1,0}(M))$ is referred to as *inextensible* if it has no extension. If N is an extension of M and Θ is a contact form on N then $(C(N), F_\Theta)$ is an extension of $(C(M), F_\theta)$ where $\theta = \iota^*\Theta$. By Proposition 3 below, $(C(\partial\Omega_0), F_{\theta_0})$ is inextensible.

5.3. **Inextensible Reeb flow.** Let $c : [0, s^*) \to \mathfrak{M}$ be an inextensible smooth timelike curve parametrized by arc length (proper time). By a result in [15] if $s^* < \infty$ then c has an endpoint on the b-boundary $\dot{\mathfrak{M}}$ provided that (36) is satisfied. A precise statement and proof are given in Appendix A. As an application (of the Dodson-Sulley-Williams lemma) we demonstrate a class of smooth curves in M having an endpoint on $\dot{M}$.

Theorem 3. *Let* (M, θ) *be a strictly pseudoconvex pseudohermitian manifold and* T *its Reeb vector. Then any inextensible integral curve* $\alpha : [a, b) \to M$ *of* T *has an endpoint on the b-boundary* $\dot{M}$ *provided that* i) α *admits a lift* $\gamma : [a, b) \to \mathfrak{M}$ *satisfying* $\dot{\gamma}(t) = f(t) \left(T^{\uparrow} - S\right)_{\gamma(t)}$ $(a \leq t < b)$ *for some* C^{∞} *function* $f : [a, b) \to \mathbb{R} \setminus \{0, 1\}$ *such that* $\phi(t) = \int_a^t f(\tau)\, d\tau$ *is bounded and* ii)

$$\int_0^{s^*} \exp\left(\int_0^t \|V\|_{\beta(s)}\, ds \right) dt < \infty \tag{23}$$

where $s^* = \lim_{t \to b^-} \phi(t)$ *and* $\beta(s) = \alpha(\phi^{-1}(s))$ *while* $V \in C^{\infty}(H(M))$ *is given by* (10)*. In particular when* (M, θ) *is pseudo-Einstein the same conclusion holds if* (ii) *is replaced by*

$$\int_0^{s^*} \exp\left(\int_0^t \|\nabla^H \rho\|_{\beta(s)}\, ds \right) dt < \infty \tag{24}$$

where ρ *is the pseudohermitian scalar curvature of* θ*. Also if* $M = S^{2n+1}$ (*endowed with the canonical contact form*) *then* (ii) *is equivalent to* $s^* < \infty$*.*

Proof. Let $c(s) = \gamma(\phi^{-1}(s))$ for any $0 \leq s < s^*$ so that $c : [0, s^*) \to \mathfrak{M}$ is an inextensible timelike curve, parametrized by arc length, with s^* finite. Then (by Lemma 1)

$$(D_{\dot{\gamma}} \dot{\gamma})_{\gamma(t)} = f'(s) \left(T^{\uparrow} - S\right)_{\gamma(t)} + f(t)^2 V^{\uparrow}_{\gamma(t)} ,$$

$$(D_{\dot{c}} \dot{c})_{c(\phi(t))} = f(t)^{-2} \left[(D_{\dot{\gamma}} \dot{\gamma})_{\gamma(t)} - f'(t) f(t)^{-1}\, \dot{\gamma}(t) \right] ,$$

hence $\|D_{\dot{c}} \dot{c}\|_{c(\phi(t))} = \|V\|_{\alpha(t)}$ for any $a \leq t < b$ and Lemma 4 yields $c(s^*-) \in \dot{\mathfrak{M}}$ hence (by Corollary 1) $\beta(s^*-) \in \dot{M}$.

Moreover if (M, θ) is pseudo-Einstein then (cf. [19], p. 298) $W^{\alpha}_{\alpha\overline{\mu}} = (i/2n)\, \rho_{\overline{\mu}}$ hence (by (10))

$$\|V\|^2 = 2 g_{\alpha\overline{\beta}} V^{\alpha} V^{\overline{\beta}} = C_n\, \rho_{\alpha}\, \rho^{\alpha} ,$$

where $2C_n = (2n+1)^2 / [n(n+1)(n+2)]^2$ (so that (23) and (24) are equivalent). When M is the odd dimensional sphere (carrying the standard contact form) the pseudohermitian scalar curvature is constant. Q.e.d.

Example 5. (*Example* 2 *continued*) Let (V, g) be a harmonic Riemannian manifold and $M_{\epsilon} = \partial\, T^{*\epsilon} V$ with the contact form θ_{ϵ}. Let T_{ϵ} be the Reeb vector of $(M_{\epsilon}, \theta_{\epsilon})$ and $\alpha : [a, b) \to M_{\epsilon}$ an inextensible integral curve of T_ϵ admitting a lift $\gamma : [a,b) \to C(M_\epsilon)$ as in Theorem 3 and satisfying (24). Then α has an endpoint on $\dot{M}_\epsilon$. Given a C^∞ function $f : [a,b) \to \mathbb{R} \setminus \{0,1\}$ the ODE system $\dot\gamma(t) = f(t)\left(T^\uparrow - S\right)_{\gamma(t)}$ admits solutions by standard existence theorems in ODE theory and their projections on M_ϵ are integral curves of T_ϵ. On the other hand integral curves of T_ϵ are chains (cf. [54], p. 393) (and therefore projections of null geodesics) thus exhibiting chains which are simultaneously projections of timelike curves in $C(M_\epsilon)$. $\square$

5.4. **b-Boundary *versus* topological boundary.** By a result in [51] for any open set $\mathcal{U} \subset \mathfrak{M}$ with compact closure one has $\partial\mathcal{U} \subset \dot{\mathcal{U}}$. The following CR analog is also true.

Proposition 2. *Let $U \subset M$ be an open subset of a strictly pseudoconvex pseudohermitian manifold (M,θ) such that its closure U^c is compact in M. Then the topological boundary ∂U is contained in the b-boundary $\dot U$.*

Proof. Let $x \in \partial U = U^c \setminus \mathring{U}$ so that $x \notin U$ by the openness of U. In particular there is a curve $\alpha : [0,1) \to U$ with endpoint x. Let $u_0 \in p^{-1}(\alpha(0))$ and let $\alpha^\uparrow : [0,1) \to O^+(\mathfrak{M})$ be the unique $\Gamma(\sigma)$-horizontal lift of α issuing at u_0. Then $\alpha^\uparrow$ has an endpoint $u_1 = \lim_{t\to 1^-} \alpha^\uparrow(t) \in \overline{p}^{-1}(x) \subset \overline{O^+(\pi^{-1}(U))}$ because U^c is compact and $\overline{p}$ is an open map. Consequently there is an integer $n_0 \geq 1$ such that $\alpha^\uparrow$ lies in the ball $B_{\overline{d}_\gamma}(u_1\,,\,1/n_0) \subset \overline{O^+(\pi^{-1}(U))}$. Let us join u_1 to some point on $\alpha^\uparrow$ by a minimizing geodesic ρ of finite length. Then $\pi\circ\rho$ is a b-incomplete curve in U with endpoint x hence $x \in \dot U$. Q.e.d.

A natural question in [51] is whether (or under which assumptions) $\partial\mathcal{U} = \dot{\mathcal{U}}$. By a result in [51] if $\mathcal{U} \subset \mathfrak{M}$ is an open subset such that i) $\mathcal{U}^c$ is compact and ii) $\mathcal{U}^c$ contains no trapped[9] null geodesic then $\partial\mathcal{U} = \dot{\mathcal{U}}$. The CR analog to this situation would be to consider an open subset $U \subset M$ such that i) U^c is compact and ii) U^c contains no trapped chain. The result in [51] doesn't readily apply to $\mathcal{U} = \pi^{-1}(U)$. Indeed $\mathcal{U}^c$ is compact (because S^1 is compact) yet $\mathcal{U}^c = \pi^{-1}(U^c)$ hence $\mathcal{U}^c$ is a saturated[10] set imprisoning all fibres of π over points in U^c (which are null-geodesics of F_θ).

5.5. **b-Boundary points and horizontal curves.** The following result is a CR analog to Theorem 4.2 in [50], p. 276. The proof however mimics that in [14], p. 461-462.

[9] In the sense of [51], p. 51.

[10] A union of leaves of the vertical foliation (tangent to S).

Theorem 4. *Let $\overline{x} \in \dot{M}$ be the point determined by the Cauchy sequence $\{v_\nu\}_{\nu \geq 1} \subset O^+(\mathfrak{M})$ i.e. $\overline{x} = \mathrm{orb}_G(\overline{v})$ where $\overline{v} = \lim_{\nu\to\infty} v_\nu \in \overline{O^+(\mathfrak{M})}$. There is a Cauchy sequence $\{u_\nu\}_{\nu\geq 1}$ lying on a $\Gamma(\sigma)$-horizontal curve in $O^+(\mathfrak{M})$ and determining the same b-boundary point $\overline{x}$.*

Proof. There is a curve $\gamma : [0,1) \to O^+(\mathfrak{M})$ such that $\gamma(t_\nu) = v_\nu$ for some $t_\nu \in [0,1)$ and any $\nu \geq 1$. There is no $x \in M$ such that $\gamma(t) \in p^{-1}(x)$ for all $0 \leq t < 1$. Indeed if it were $\gamma \subset p^{-1}(x_0)$ for some $x_0 \in M$ then $\{v_\nu\}_{\nu\geq 1}$ would have a limit there (by (iii) in Theorem 1 the fibre $p^{-1}(x)$ is complete) instead of $\overline{O^+(\mathfrak{M})} \setminus O^+(\mathfrak{M})$. Let us set $\alpha = p\circ\gamma : [0,1) \to M$ (an inextensible curve). Let $\alpha^\uparrow : [0,1) \to O^+(\mathfrak{M})$ be the unique $\Gamma(\sigma)$-horizontal lift of α issuing at $\alpha^\uparrow(0) = v_1$ and let us consider the sequence $\alpha^\uparrow(t_\nu) \in O^+(\mathfrak{M})$ projecting on $x_\nu = p(v_\nu) \in M$. As $G = \mathrm{O}^+(1, m-1) \times S^1$ acts transitively on fibres for each $\nu \geq 1$ there is $a_\nu \in G$ such that $R_{a_\nu}(\alpha^\uparrow(t_\nu)) = v_\nu$ (actually $a_1 = e$). Let $v_\infty = \lim_{\nu\to\infty} v_\nu \in \overline{O^+(\mathfrak{M})} \setminus O^+(\mathfrak{M})$. Also (as $\overline{O^+(\mathfrak{M})}$ is complete) the curve $\alpha^\uparrow$ has an endpoint $u_\infty = \lim_{t\to 1^-} \alpha^\uparrow(t) \in \overline{O^+(\mathfrak{M})} \setminus O^+(\mathfrak{M})$ such that $p(u_\infty) = p(v_\infty) = \overline{x} \in \dot{M}$. Let then $a_\infty \in G$ such that $R_{a_\infty}(u_\infty) = v_\infty$.

By a result of B.G. Schmidt, [52], $O^+(1, m-1)$ is closed in $\mathrm{GL}^+(m, \mathbb{R})$ and then G is closed in $\mathrm{GL}^+(m,\mathbb{R}) \times S^1$ hence (by the continuity of the action) $\lim_{\nu\to\infty} a_\nu = a_\infty$ in G. Let $\alpha^* : [0,1) \to O^+(\mathfrak{M})$ be the unique $\Gamma(\sigma)$-horizontal lift of α issuing at $\alpha^*(0) = R_{a_\infty}(v_1)$ and let us consider the sequence $u_\nu = \alpha^*(t_\nu) \in O^+(\mathfrak{M})$.

By the general theory of connections in principal bundles (cf. e.g. [37], Vol. I) right translations by elements of the structure group map horizontal curves into horizontal curves hence $R_{a_\infty} \circ \alpha^\uparrow$ is a horizontal curve issuing at the same point as α^* and then $\alpha^* = R_{a_\infty} \circ \alpha^\uparrow$. In particular $u_\nu = R_{a_\infty}(\alpha^\uparrow(t_\nu))$ for any $\nu \geq 1$.

As in the proof of Lemma 2 there is a constant $\beta(a_\infty) > 0$ such that

$$d_\gamma(R_{a_\infty}(u)\,,\, R_{a_\infty}(v)) \leq \beta(a_\infty)\, d_\gamma(u,v), \quad u, v \in O^+(\mathfrak{M}).$$

Also (again by the continuity of the action of G on $\overline{O^+(\mathfrak{M})}$) one has

$$\lim_{\nu\to\infty} R_{a_\nu^{-1}} v_\nu = \overline{R}_{a_\infty^{-1}} v_\infty$$

hence

$$d_\gamma(u_\nu\,,\, u_\mu) = d_\gamma(R_{a_\infty}\alpha^\uparrow(t_\nu)\,,\, R_{a_\infty}\alpha^\uparrow(t_\mu)) \leq$$

$$\leq \beta(a_\infty)\, d_\gamma(\alpha^\uparrow(t_\nu)\,,\, \alpha^\uparrow(t_\mu)) = \beta(a_\infty)\, \overline{d}_\gamma(R_{a_\nu^{-1}}(v_\nu)\,,\, R_{a_\mu^{-1}}(v_\mu))$$

implying that $\{u_\nu\}_{\nu\geq 1}$ is a Cauchy sequence with the same limit $v_\infty \in \overline{O^+(\mathfrak{M})} \setminus O^+(\mathfrak{M})$ as $\{v_\nu\}_{\nu\geq 1}$. Q.e.d.

Example 6. (*Example* 3 *continued*) The Fefferman metric of $(\mathbb{H}_1, \theta_0)$ is given by [with respect to the (global) coordinates $(x^\alpha) \equiv (x, y, t, \gamma)$ on $C(\mathbb{H}_1)$]

$$[F_{\alpha\beta}]_{0\le\alpha,\beta\le 3} = \begin{pmatrix} 2 & 0 & 0 & -ay \\ 0 & 2 & 0 & ax \\ 0 & 0 & 0 & a/2 \\ -ay & ax & a/2 & 0 \end{pmatrix} \tag{25}$$

where $a = 2/3$. If $F_{\alpha\beta}F^{\beta\gamma} = \delta_\alpha^\gamma$ then

$$\left[F^{\alpha\beta}\right]_{0\le\alpha,\beta\le 3} = \begin{pmatrix} 1/2 & 0 & y & 0 \\ 0 & 1/2 & -x & 0 \\ y & -x & 2|z|^2 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix} \tag{26}$$

where $z = x + iy$. As a consequence of (25) the only nonvanishing Christoffel symbols $|\alpha\beta, \gamma| = \frac{1}{2}(F_{\alpha\gamma|\beta} + F_{\beta\gamma|\alpha} - F_{\alpha\beta|\gamma})$ are

$$|03, 1| = |30, 1| = a, \quad |13, 0| = |31, 0| = -a.$$

Consequently the nonzero Christoffel symbols $\left| \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right| = F^{\alpha\lambda}|\beta\gamma, \lambda|$ are

$$\left| \begin{matrix} 0 \\ 13 \end{matrix} \right| = \left| \begin{matrix} 0 \\ 31 \end{matrix} \right| = -a/2\,, \quad \left| \begin{matrix} 2 \\ 13 \end{matrix} \right| = \left| \begin{matrix} 2 \\ 31 \end{matrix} \right| = -ay,$$

$$\left| \begin{matrix} 1 \\ 03 \end{matrix} \right| = \left| \begin{matrix} 1 \\ 30 \end{matrix} \right| = a/2, \quad \left| \begin{matrix} 2 \\ 03 \end{matrix} \right| = \left| \begin{matrix} 2 \\ 30 \end{matrix} \right| = -ax,$$

so that the equations of geodesics of $(C(\mathbb{H}_1), F_{\theta_0})$

$$\frac{d^2 C^\alpha}{ds^2} + \left| \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right| \frac{dC^\beta}{ds}\frac{dC^\gamma}{ds} = 0$$

read

$$\frac{d^2 z}{ds^2} + iap\frac{dz}{ds} = 0, \tag{27}$$

$$\frac{d^2 t}{ds^2} - ap\frac{d|z|^2}{ds} = 0, \quad \gamma(s) = ps + q, \quad p, q \in \mathbb{R}. \tag{28}$$

We obtain

Proposition 3. i) *The geodesics of* $(C(\mathbb{H}_1), F_{\theta_0})$ *are either*

$$z(s) = \lambda \, \exp(-i\, aps) + \frac{\mu}{iap}\,, \quad \lambda, \mu \in \mathbb{C}, \tag{29}$$

$$t(s) = \left(\ell + ap|\lambda|^2 + \frac{|\mu|^2}{ap}\right) s + \tag{30}$$

$$+m - \frac{1}{ap}\left[\lambda\overline{\mu}e^{-i\,aps} + \overline{\lambda}\mu e^{i\,aps}\right], \quad \ell,\, m \in \mathbb{R},$$

$$\gamma(s) = ps + q, \quad p, q \in \mathbb{R}, \tag{31}$$

provided that $p \neq 0$ *or*

$$z(s) = \lambda s + \mu, \;\; t(s) = \ell s + m, \;\; \gamma(s) = q, \quad \lambda, \mu \in \mathbb{C}, \; \ell, m, q \in \mathbb{R}, \tag{32}$$

when $p = 0$. ii) F_{θ_0} *is geodesically complete.* iii) *Each geodesic* (32) *is* σ*-horizontal.* iv) *Let* $\ell_0 = -ap|\lambda|^2 - (1/ap)|\mu|^2$ $(p \neq 0)$. *The geodesics* (29)-(31) *are spacelike when* $\ell > \ell_0$, *timelike when* $\ell < \ell_0$, *or null when* $\ell = \ell_0$. *If* $\ell = \ell_0$ *then* (29)-(31) *projects on a family of closed chains in* $\partial\Omega_0 \approx \mathbb{H}_1$. v) (32) *is a family of geodesics intersecting each fibre of* $\pi : C(\mathbb{H}_1) \to \mathbb{H}_1$ *once. Each geodesic in the family is either spacelike* $(\lambda \neq 0)$ *or null* $(\lambda = 0)$.

Proof. (29)-(30) is the general solution to (27)-(28). In particular $(C(\partial\Omega_0), F_{\theta_0})$ is inextensible. Statement (iii) follows from $\sigma = (a/2)\, d\gamma$. For each geodesic $C(t) \in C(\mathbb{H}_1)$ one has $F_{\alpha\beta}\dot{C}^\alpha\dot{C}^\beta = 2|\dot{z}|^2 + 2ap\,\mathrm{Im}(\overline{z}\dot{z}) + ap\,\dot{t}$ hence

$$F_{\alpha\beta}(C(s))\dot{C}^\alpha(s)\dot{C}^\beta(s) = a^2p^2|\lambda|^2 + |\mu|^2 + ap\ell, \quad s \in \mathbb{R}. \tag{33}$$

yielding statement (iv).

Theorem 5. *The b-boundary of* $(\mathbb{H}_1\,,\,\theta_0)$ *is empty* $(\dot{\mathbb{H}}_1 = \emptyset)$.

A *verbatim* repetition of the arguments in the proof of Lemma 2 describes $\Gamma(\sigma)$ as a connection in $\mathrm{GL}^+(4) \times S^1 \to L^+(C(\mathbb{H}_1)) \to \mathbb{H}_1$ (reducible to $\mathrm{O}^+(1,3) \times S^1 \to O^+(C(\mathbb{H}_1)) \to \mathbb{H}_1$). Let $\alpha : [0,1) \to \mathbb{H}_1$ be a smooth inextensible curve and $\alpha^\uparrow : [0,1) \to L^+(C(\mathbb{H}_1))$ its $\Gamma(\sigma)$-horizontal lift issuing at $\alpha^\uparrow(0) = u$. Let $\beta_{\alpha^\uparrow(s)} : T_{c(s)}(C(\mathbb{H}_1)) \to \Gamma_{\alpha^\uparrow(s)}$ be the Γ-horizontal lift where $c = \Pi_{L^+} \circ \alpha^\uparrow$. We set $X^* = \beta X$ for each $X \in \mathfrak{X}(C(\mathbb{H}_1))$. Then

$$\dot{\alpha}^\uparrow(s) = \frac{d\alpha^\lambda}{ds}\left(\frac{\partial}{\partial x^\lambda}\right)_{\alpha^\uparrow(s)} + \frac{d\alpha^\lambda_\mu}{ds}\left(\frac{\partial}{\partial X^\lambda_\mu}\right)_{\alpha^\uparrow(s)} =$$

$$= \frac{d\alpha^\lambda}{ds}\left(\frac{\partial}{\partial x^\lambda}\right)^*_{\alpha^\uparrow(s)} + \left\{\frac{d\alpha^\lambda_\mu}{ds} + \frac{d\alpha^\nu}{ds}\left|\begin{array}{c}\lambda\\ \nu\sigma\end{array}\right|(c(s))\alpha^\sigma_\mu(s)\right\}\left(\frac{\partial}{\partial X^\lambda_\mu}\right)_{\alpha^\uparrow(s)}$$

where $\alpha^\lambda = x^\lambda \circ \alpha^\uparrow$ and $\alpha^\lambda_\mu = X^\lambda_\mu \circ \alpha^\uparrow$. Moreover $\Gamma(\sigma) \subset \Gamma$ yields

$$\frac{d\alpha^\lambda_\mu}{ds} + \frac{d\alpha^\nu}{ds}\left(\left|\begin{array}{c}\lambda\\ \nu\sigma\end{array}\right| \circ c\right)\alpha^\sigma_\mu = 0. \tag{34}$$

In particular

$$\Gamma(\sigma)_{\alpha^\uparrow(s)} \ni \dot{\alpha}^\uparrow(s) = \sum_{j=0}^{2} \frac{d\alpha^j}{ds}(s) \left(\frac{\partial}{\partial x^j}\right)^\uparrow_{\alpha^\uparrow(s)} + \frac{d\gamma}{ds}(s) \left(\frac{\partial}{\partial \gamma}\right)^*_{\alpha^\uparrow(s)}$$

and $\mathrm{Ker}(\sigma)$ is the span of $\{\partial/\partial x,\ \partial/\partial y,\ \partial/\partial t\}$ hence $d\gamma/ds = 0$. Thus $\gamma(s) = m$ with $m \in \mathbb{R}$ and (34) reads

$$\frac{d}{ds}\left(\alpha^0_\mu + i\,\alpha^1_\mu\right) = \frac{a}{2i}\frac{dz}{ds}\,\alpha^3_\mu\,, \quad \frac{d\alpha^2_\mu}{ds} = \frac{a}{2}\frac{d|z|^2}{ds}\,\alpha^3_\mu\,, \quad \frac{d\alpha^3_\mu}{ds} = 0. \tag{35}$$

The general solution to (35) is

$$\alpha^0_\mu(s) = p_\mu + \frac{a}{2}\,t_\mu\,y(s)\,, \quad \alpha^1_\mu(s) = q_\mu - \frac{a}{2}\,t_\mu\,x(s)\,,$$

$$\alpha^2_\mu(s) = r_\mu + \frac{a}{2}\,t_\mu\,|z(s)|^2\,, \quad \alpha^3_\mu(s) = t_\mu\,, \quad p_\mu\,,\, q_\mu\,,\, r_\mu\,,\, t_\mu \in \mathbb{R}.$$

Together with Theorem 2 this gives

Lemma 3. *The b-boundary $\dot{\mathbb{H}}_1$ consists of the endpoints of all inextensible curves $\alpha = (x,y,t) : [0,1) \to \mathbb{H}_1$ of finite Euclidean length i.e. $\int_0^1 \left[\dot{x}^2 + \dot{y}^2 + \dot{t}^2\right]^{1/2} ds$ is convergent.*

Proof. There is a constant degenerate matrix $\alpha_0 \in \mathbb{R}^{16}$ such that

$$\left[\alpha^\lambda_\mu(s)\right] = \frac{a}{2}\begin{pmatrix} t_0 y(s) & t_1 y(s) & t_2 y(s) & t_3 y(s) \\ -t_0 x(s) & -t_1 x(s) & -t_2 x(s) & -t_3 x(s) \\ t_0|z(s)|^2 & t_1|z(s)|^2 & t_2|z(s)|^2 & t_3|z(s)|^2 \\ (2/a)t_0 & (2/a)t_1 & (2/a)t_2 & (2/a)t_3 \end{pmatrix} + \alpha_0\,.$$

We may assume the connected component $L^+(C(\mathbb{H}_1))$ is chosen such that for each $c_0 \in \pi^{-1}(\alpha(0))$ one has

$$u_0 = (c_0, \{(\partial/\partial x^\mu)_{c_0} : 0 \le \mu \le 4\}) \in L^+(C(\mathbb{H}_1)).$$

We wish to describe curves $\alpha : [0,1) \to \mathbb{H}_1$ having finite b-length. Let $\alpha^\uparrow : [0,1) \to L^+(C(\mathbb{H}_1))$ be the $\Gamma(\sigma)$-horizontal lift of α issuing at u_0. Then $X^\lambda_\mu(u_0) = \delta^\lambda_\mu$ and $\alpha^\uparrow(0) = u_0$ yield

$$\alpha_0 = \begin{pmatrix} 1 & 0 & 0 & -(a/2)y_0 \\ 0 & 1 & 0 & (a/2)x_0 \\ 0 & 0 & 1 & -(a/2)|z_0|^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t_\mu = \delta^3_\mu\,,$$

where $x_0 = x(0)$, $y_0 = y(0)$ and $z_0 = x_0 + iy_0$. Therefore

$$\left[\alpha^\lambda_\mu\right] = \begin{pmatrix} 1 & 0 & 0 & (a/2)(y - y_0) \\ 0 & 1 & 0 & -(a/2)(x - x_0) \\ 0 & 0 & 1 & (a/2)\left[|z|^2 - |z_0|^2\right] \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$[\beta_\mu^\lambda] \equiv [\alpha_\mu^\lambda]^{-1} = \begin{pmatrix} 1 & 0 & 0 & -(a/2)(y-y_0) \\ 0 & 1 & 0 & (a/2)(x-x_0) \\ 0 & 0 & 1 & -(a/2)\left[|z|^2 - |z_0|^2\right] \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally $\beta_\mu^\lambda(s) e_\lambda = \alpha^\uparrow(s)^{-1}\left(\partial/\partial x^\mu\right)_{c(s)}$ yields

$$\int_0^1 \|\alpha^\uparrow(s)^{-1}\dot{c}(s)\|\, ds = \int_0^1 \left[\dot{x}(s)^2 + \dot{y}(s)^2 + \dot{t}(s)^2\right]^{\frac{1}{2}} ds.$$

Lemma 3 is proved. Then Theorem 5 follows from Lemma 3 and the fact that curves $\alpha : [0,1) \to \mathbb{H}_1$ of finite Euclidean length are actually continuously extendible at $s = 1$. For given a sequence $\{s_\nu\}_{\nu\geq 1} \subset [0,1)$ such that $s_\nu \to 1$ as $\nu \to \infty$

$$d_0(\alpha(s_\nu), \alpha(s_\mu)) \leq \left| \int_{s_\nu}^{s_\mu} (\dot{x}^2 + \dot{y}^2 + \dot{t}^2)^{1/2}\, ds \right|$$

(where d_0 is the Euclidean distance on $\mathbb{R}^3$) hence $\{\alpha(s_\nu)\}_{\nu\geq 1}$ is Cauchy in $(\mathbb{R}^3, d_0)$ and then convergent. $\square$

6. Comments and open problems

Several other CR and pseudohermitian analogs to B. Schmidt's construction (of a b-completion and b-boundary) have been suggested by the Reviewers. For every nondegenerate CR manifold, on which a contact form θ has been fixed, the Tanaka-Webster connection ∇ of (M, θ) gives rise to a connection-distribution Γ^∇ in the principal bundle $\mathrm{Gl}(2n+2, \mathbb{R}) \to L(M) \to M$. As $\nabla g_\theta = 0$ the connection Γ^∇ is reducible to the bundle of g_θ-orthonormal frames $\mathrm{O}(2r+1, 2s) \to O(M, g_\theta) \to M$. Only a linear connection on M is needed (cf. e.g. [16]) to build a bundle completion, so both principal bundles $L(M)$ and $O(M, g_\theta)$ are suitable for repeating the construction in [50]. For instance let $\omega^\nabla \in C^\infty(T^*(L(M)) \otimes \mathfrak{gl}(2n+1, \mathbb{R}))$ be the connection 1-form corresponding to Γ^∇ and let us set

$$g_u^\nabla(X, Y) = \omega_u^\nabla(X) \cdot \omega_u^\nabla(Y) + \eta_{M,u}(X) \cdot \eta_{M,u}(Y)$$

for any $X, Y \in T_u(L(M))$ and $u \in L(M)$. Here $\eta_M \in C^\infty(T^*(L(M)) \otimes \mathbb{R}^{2n+1})$ is the canonical 1-form and dots indicate Euclidean inner products in $\mathbb{R}^{(2n+1)^2}$ and $\mathbb{R}^{2n+1}$. Then g^∇ is a Riemannian metric on $L(M)$ and one may consider the Cauchy completion $\overline{L^+(M)}$ of the metric space $(L^+(M), d^\nabla)$ where d^∇ is the distance function associated to g^∇. Comparing $\overline{L^+(M)}$ and $\overline{L^+(\mathfrak{M})}$ is an open problem. For instance, let $s : M \to \mathfrak{M}$ be a global section. Such s exists when

M is a real hypersurface in $\mathbb{C}^{n+1}$ [for then $\mathfrak{M} \approx M \times S^1$ (cf. e.g. [42]) and one may set $s(x) = (x, \mathbf{1})$ for any $x \in M$]. Then the map $f : L(M) \to L(\mathfrak{M})$ given by $f(u) = (s(x), \{v_i^{\uparrow}, S_{s(x)} : 1 \leq i \leq 2n+1\})$ for any $u = (x, \{v_i : 1 \leq i \leq 2n+1\}) \in L(M)$ and any $x \in M$, is an immersion and one ought to compute the $(0,2)$-tensor field $f^* g - g^{\nabla}$. Here $v_i^{\uparrow} \in \operatorname{Ker}(\sigma_{s(x)})$ and $(d_{s(x)}\pi) v_i^{\uparrow} = v_i$. As emphasized above, both $\overline{L^+(M)}$ and $\overline{L^+(\mathfrak{M})}$ depend upon the choice of θ. The behavior of the completions (and corresponding boundaries) with respect to a transformation $\hat{\theta} = \pm e^u \theta$ (with $u \in C^\infty(M)$) of the contact form is unknown.

The Carnot-Carathéodory metric d_H may fail to be complete, in general, yet completeness of d_H is relevant in a number of geometric problems (cf. [55] and [4]). The lack of completeness on d_H prompts the use of the Cauchy completion $\overline{M}$ of the metric space (M, d_H). By analogy to the work of K. Nomizu & H. Ozeki, [44] (that for any Riemannian metric there is a conformal transformation such that the resulting metric is complete) one may ask whether $u \in C^\infty(M)$ exists such that any of d on $L^+(\mathfrak{M})$, d^{∇} on $L^+(M)$, or d_H on M (built in terms of $\hat{\theta}$ above) be complete.

Another approach to completions and boundaries, as mentioned in § 1, may rely on the canonical Cartan geometry of the given nondegenerate CR manifold, which is also related to the conformal structure determined by the Fefferman metric F_θ, A. Čap & A.R. Gover, [12]. Cf. also C. Frances, [23].

Example 1 (used in this paper only for $\delta = 0$) is meant to suggest possible applications of our constructions (of $\overline{\partial \Omega_\delta}$ and corresponding boundary) and (by including Fefferman's example $\delta = 1$) that chains may play a fundamental role in understanding the nature of boundary points. The same comment applies to Examples 2 and 5 (on boundaries of Grauert tubes) due to the work by M.B. Stenzel, [54] (relating chains to integral curves of Reeb vector fields).

Recent discussion (cf. A. Kempf, [36], T. Kopf, [39]-[40], A. Prain, [47]) of the possibly discrete nature of spacetime (eventually springing from considerations within quantum field theory) relies on the use of spectra of naturally defined differential operators (e.g. the Dirac operator associated to a fermionic Weyl spinor field, [40]). Comments based on the analogy to spectral geometry on compact Riemannian manifolds (cf. e.g. [36]) remain rather speculative *vis-a-vis* to the non-compactness of spacetime and the hyperbolic (rather than elliptic) nature of the Laplace-Beltrami operator of the given Lorentzian metric. Non-compactness of $\mathfrak{M} = C(M)$ and non-ellipticity of $\Box$ (the wave operator i.e. the Laplace-Beltrami operator of F_θ) are, very much the same, an obstacle towards a discrete description of $(\mathfrak{M}, F_\theta)$ yet $\Box$ is related to the sublaplacian Δ_b (by a result of J.M. Lee, [41], the pushforward of $\Box$ is precisely Δ_b i.e. $\pi_* \Box = \Delta_b$) which again isn't elliptic, yet it is at least subelliptic (with a loss of $\frac{1}{2}$ derivative) and hence hypoelliptic (a feature enjoyed by elliptic operators and highly exploited in subelliptic theory, cf. e.g. [8]). Again the information that Δ_b has a discrete spectrum is available only on compact strictly pseudoconvex CR manifolds [when $\mathfrak{M}$ is compact too and thus (by our discussion in § 3) it is neither causal nor chronological]. Coming to a possible sampling theory of spacetimes (cf. [36], p. 8), in order to build a theory anchored into Riemannian geometry, one may attempt to apply M. Kanai's discrete approach (cf. [35]) to the Riemannian manifold $(O^+(\mathfrak{M}), \gamma)$. Lack of completeness is again relevant to employing rough isometries (as in [35]) and may prompt the use of b-completions and b-boundaries.

Appendix A. The Dodson-Sulley-Williams lemma

The work by C.T.J. Dodson et al., [15], gives a sufficient condition that an inextensible timelike curve has an endpoint on the b-boundary. This is a variant of a result by B.G. Schmidt, [50], in the presence of an assumption of the type "boundedness of acceleration" inspired by work of F. Rosso, [48]. Also [15] adjusts some imprecisions in [48] yet the proof applies only under the additional condition that the x^0-component of the velocity in Minkowski space is almost everywhere non unit. We restate the result in [15] (referred to through this paper as the *Dodson-Sulley-Williams lemma*) as it applies to the pseudohermitian context and give a proof for the sake of completeness.

Lemma 4. ([15], p. 193) *Let $(M, T_{1,0}(M), \theta)$ be a strictly pseudoconvex pseudohermitian manifold and $c : [0, s^*) \to \mathfrak{M}$ an inextensible timelike curve parametrized by arc length. Let $u(s) = (c(s), \{X_{j,c(s)} : 1 \le j \le m\}) \in O^+(\mathfrak{M})$ be a parallel frame and let $\dot{c}(s) = \xi^j(s) X_{j,c(s)}$ be the components of the tangent vector $\dot{c}$. Then $D_{\dot{c}}\dot{c}$ is either spacelike or null. If* i) *the set $\{s \in [0, s^*) : \xi^1(s) = 1\}$ has Lebesgue measure zero,* ii) $s^* < \infty$ *and* iii)

$$\int_0^{s^*} \exp\left(\int_0^t \|D_{\dot{c}}\dot{c}\| \, ds \right) dt < \infty, \tag{36}$$

then c has an endpoint on $\dot{\mathfrak{M}}$.

The frame $u(t)$ has been chosen such that $(D_{\dot{c}} X_j)_{c(s)} = 0$ i.e. $\dot{u}(s) \in \Gamma_{u(s)}$ for any $0 \le s \le s^*$. It gives a linear isometry among $T_{c(s)}(\mathfrak{M})$

and the Minkowski space $\mathbb{M}$ [i.e. $\mathbb{R}^m$ with the quadratic form $-(x^0)^2 + \sum_{a=1}^{m-1}(x^a)^2$]. If $\xi(s) = (\xi^1(s), \cdots, \xi^m(s))$ are the components of $\dot{c}(s)$ then

$$L(c^\uparrow) = \int_0^{s^*} \|c^\uparrow(s)^{-1}\dot{c}(s)\|\, ds = \int_0^{s^*} \|\xi(s)\|\, ds$$

where $\|\xi\|$ is the Euclidean norm of $\xi \in \mathbb{R}^m$. Let us set $\xi' = (\xi^2, \cdots, \xi^m)$ so that $\xi = (\xi^1, \xi')$. Then

$$1 = -F_\theta(\dot{c}\,,\,\dot{c}) = -\epsilon_j \delta_{jk}\xi^j\xi^k = \left(\xi^1\right)^2 - \|\xi'\|^2$$

where $\|\xi'\|$ is the Euclidean norm in $\mathbb{R}^{m-1}$. This is exploited in two ways i.e. first it yields $|\xi^1(s)| \geq 1$ hence

$$\phi(s) = \int_0^s \xi^1(\sigma)\, d\sigma$$

is a new parameter (for $\dot{\phi} = \xi^1$ so that $\dot{\phi}$ has constant sign) and second it yields $2\,|\xi^1|^2 > \|\xi\|^2$ so that one may conduct the following estimates

$$\int_0^{s^*} |\xi^1|\, ds \leq L(c^\uparrow) = \int_0^{s^*} \|\xi\|\, ds \leq \int_0^{s^*} \left(2(\xi^1)^2\right)^{1/2} ds = \sqrt{2}\int_0^{s^*} |\xi^1|\, ds$$

hence $L(c^\uparrow) < \infty$ if and only if $\int_0^{s^*} |\xi^1|\, ds < \infty$. Combining that with a result in [50] one has $c(s^*-) = \lim_{s\to(s^*)^-} c(s) \in \dot{\mathfrak{M}}$ if and only if ϕ is bounded. We claim that

$$\ddot{\phi}^2 \leq \left(\dot{\phi}^2 - 1\right) F_\theta(D_{\dot{c}}\dot{c}\,,\, D_{\dot{c}}\dot{c}). \tag{37}$$

In particular for each value of the parameter $(D_{\dot{c}}\dot{c})_{c(s)}$ is either spacelike or null[11]. To prove (37) let $E_{c(s)}$ be the span of $\{X_{a,c(s)} : 2 \leq a \leq m\}$. and let $P_{c(s)} : T_{c(s)}(\mathfrak{M}) \to E_{c(s)}$ be the projection. As $(D_{\dot{c}}X_1)_{c(s)} = 0$

$$D_{\dot{c}}\dot{c} = D_{\dot{c}}\left[P\dot{c} - F_\theta(\dot{c}\,,\,X_1)X_1\right] = D_{\dot{c}}P\dot{c} - \frac{d}{ds}\left\{F_\theta(\dot{c}, X_1)\right\}X_1$$

or

$$D_{\dot{c}}\dot{c} = D_{\dot{c}}P\dot{c} + \ddot{\phi}\,X_1\,. \tag{38}$$

Next $DF_\theta = 0$ and again $D_{\dot{c}}X_1 = 0$ yield

$$\frac{d}{ds}\left\{F_\theta(P\dot{c}\,,\,X_1)\right\} = F_\theta(D_{\dot{c}}P\dot{c}\,,\,X_1)$$

hence (as X_1 is orthogonal to E)

$$F_\theta(D_{\dot{c}}P\dot{c}\,,\,X_1) = 0 \tag{39}$$

[11]When $D_{\dot{c}}\dot{c}$ is null assumption (iii) is equivalent to (ii) (and a slight modification of the proof yields Dodson-Sulley-Williams lemma).

i.e. $D_{\dot c} P\dot c \in E$. Then (38)-(39) imply

$$F_\theta(D_{\dot c}\dot c\,,\, D_{\dot c}\dot c) = \|D_{\dot c}P\dot c\|^2 - \ddot\phi^2\,. \tag{40}$$

The norm notation in the right hand side of (40) is justified by the fact that F_θ is positive definite on E (one should however postpone writing $\|D_{\dot c}\dot c\|^2$ until the end of the proof of (37)). Also Cauchy-Schwartz inequality holds for the restriction of F_θ to E hence

$$|F_\theta(D_{\dot c}P\dot c\,,\, P\dot c)| \le \|D_{\dot c}P\dot c\|\,\|P\dot c\| = \quad \text{(by (40))}$$

$$= \left[F_\theta(D_{\dot c}\dot c\,,\, D_{\dot c}\dot c) + \ddot\phi^2\right]^{1/2}\,\|P\dot c\|$$

or

$$F_\theta(D_{\dot c}P\dot c\,,\, P\dot c)^2 \le \left[F_\theta(D_{\dot c}\dot c\,,\, D_{\dot c}\dot c) + \ddot\phi^2\right]\,\|P\dot c\|^2. \tag{41}$$

On the other hand

$$-1 = F_\theta(\dot c\,,\,\dot c) = \|P\dot c\|^2 - F_\theta(\dot c\,,\, X_1)^2$$

or

$$\|P\dot c\|^2 = -1 + \dot\phi^2. \tag{42}$$

Also (again due to $DF_\theta = 0$)

$$2\,F_\theta(D_{\dot c}P\dot c\,,\, P\dot c) = \frac{d}{ds}\left\{\|P\dot c\|^2\right\}$$

i.e. (by (42))

$$F_\theta(D_{\dot c}P\dot c\,,\, P\dot c) = \dot\phi\ddot\phi. \tag{43}$$

Finally one may substitute from (43) into (41)

$$\left(\dot\phi\ddot\phi\right)^2 \le \left[F_\theta(D_{\dot c}\,,\, D_{\dot c}\dot c) + \ddot\phi^2\right]\left(\dot\phi^2 - 1\right)$$

yielding (37). At this point one may complete the proof of Lemma 4 as follows

$$\frac{\phi(s^*)}{\dot\phi(0)} = \frac{1}{\dot\phi(0)}\int_0^{s^*}\xi^1(t)\,dt = \int_0^{s^*}\frac{\dot\phi(t)}{\dot\phi(0)}\,dt =$$

$$= \int_0^{s^*}\exp\left(\log\frac{\dot\phi(t)}{\dot\phi(0)}\right)dt \le \int_0^{s^*}\exp\left|\log\frac{\dot\phi(t)}{\dot\phi(0)}\right|dt =$$

$$= \int_0^{s^*}\exp\left|\int_0^t\frac{\ddot\phi(s)}{\dot\phi(s)}\,ds\right|dt \le \int_0^{s^*}\exp\left(\int_0^t\left|\frac{\ddot\phi(s)}{\dot\phi(s)}\right|ds\right)dt \le$$

$$\leq \int_0^{s^*} \exp\left(\int_0^t \frac{\left|\ddot{\phi}(s)\right|}{\left(\dot{\phi}(s)^2 - 1\right)^{1/2}} ds\right) dt \leq \quad \text{(by (37))}$$

$$\leq \int_0^{s^*} \exp\left(\int_0^t \|D_{\dot{c}}\dot{c}\| \, ds\right) dt$$

hence (under the assumptions of Lemma 4) $\phi(s^*)$ is bounded so that $c(s^*-) \in \dot{\mathfrak{M}}$. Q.e.d.

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