I) Introduction[1] to CR geometry and subelliptic harmonic maps. II) Boundary values of Bergman-harmonic maps

Sorin Dragomir[2]

Abstract. We give an elementary introduction to CR and pseudohermitian geometry, starting from H. Lewy's legacy (cf. [20]) i.e. tangential Cauchy-Riemann equations on the boundary of the Siegel domain. In this context we describe fundamental objects, such as contact structures, Levi forms, the Tanaka-Webster connection and the Fefferman metric (cf. e.g. [4]). Also naturally arising Hörmander systems of vector fields, the associated sublaplacians, and J. Jost and C-J. Xu's subelliptic harmonic maps (cf. [16]), a first geometric interpretation of which is given in terms of Lorentzian geometry (cf. [2]). A second, more specialized talk - scheduled for the afternoon of the same day - is devoted to the discussion of boundary values of Bergman-harmonic maps. There we start from A. Korányi & H.M. Reimann's crucial observation (cf. [17]) that, as a consequence of Fefferman's asymptotic expansion formula (cf. [9]) for the Bergman kernel of a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$

$$K(\zeta, z) = C_\Omega |\nabla \varphi(z)|^2 \cdot \det L_\varphi(z) \cdot \Psi(\zeta, z)^{-(n+1)} + E(\zeta, z),$$

$$|E(\zeta, z)| \leq C'_\Omega \, |\Psi(\zeta, z)|^{-(n+1)+1/2} \cdot |\log |\Psi(\zeta, z)|| \, ,$$

the Kählerian geometry of the interior of Ω may be effectively related to the contact geometry of its boundary $\partial\Omega$. Then we make use of the Graham-Lee connection (cf.[13]) to derive the compatibility equations on $\partial\Omega$ satisfied by the boundary values of a Bergman-harmonic map $\Phi : \overline{\Omega} \to S$ which is C^∞ up the boundary. We are led to a geometric interpretation (cf. [5]) of Jost & Xu's subelliptic harmonic maps from an open set $U \subset \mathbb{R}^{2n-1}$ carrying a given Hörmander system of vector fields.

[1]Talks given at The State University of New Jersey Rutgers Campus at Camden, Department of Mathematical Sciences, within *A Day for Complex Analysis and Geometry*, May 3, 2013.

[2]Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Viale dell'Ateneo Lucano 10, Campus Macchia Romana, 85100 Potenza, Italy, e-mail `sorin.dragomir@unibas.it`

1. Tangential Cauchy-Riemann equations: Hans Lewy's legacy

Let us start with the Siegel domain

$$\Omega_n = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \mathrm{Im}(w) > |z|^2\}$$

and consider the Dirichlet problem for the ordinary Cauchy-Riemann system

$$\overline{\partial} F = 0 \quad \text{in} \quad \Omega_n, \tag{1}$$

$$F = f \quad \text{on} \quad \partial\Omega_n, \tag{2}$$

for some $f \in C^\infty(\partial\Omega_n, \mathbb{C})$. Let us look at the C^∞ regularity up to the boundary of the solution to (1)-(2) i.e. assume that a solution $F \in C^\infty(\overline{\Omega}_n, \mathbb{C})$ exists. If

$$\rho(z, w) = \frac{1}{2i}(w - \overline{w}) - \sum_{j=1}^n z_j \overline{z}_j$$

and $\epsilon > 0$ then $M_\epsilon = \{(z, w) \in \mathbb{C}^{n+1} : \rho(z, w) = \epsilon\}$ is a smooth real hypersurface lying in the interior of Ω_n. A complex tangent vector field

$$Z = \sum_{j=1}^n \lambda_j \frac{\partial}{\partial z_j} + \mu \frac{\partial}{\partial w}$$

is tangent to M_ϵ if and only if $Z(\rho) = 0$ i.e. $\mu = 2i \sum_j \lambda_j \overline{z}_j$. Complex vector fields, of type $(1, 0)$, which are tangent to M_ϵ are therefore of the form $Z = \sum_j \lambda_j L_j$ where

$$L_j \equiv \frac{\partial}{\partial z_j} + 2i\overline{z}_j \frac{\partial}{\partial w} .$$

As F is holomorphic in Ω_n

$$\frac{\partial F}{\partial \overline{z}_j} = 0, \quad \frac{\partial F}{\partial \overline{w}} = 0,$$

one has $\overline{L}_j(F) = 0$ and therefore along M_ϵ. Let us tend with $\epsilon \to 0$, which is the same to approach the boundary as $(z, w) \to \partial\Omega_n$. It follows that the boundary values f of F must satisfy the equations

$$\overline{L}_j f = 0. \tag{3}$$

In the language of PDEs, C^∞ regularity up to the boundary in the Dirichlet problem (1)-(2) doesn't come for free, the boundary data must satisfy certain compatibility equations along $\partial\Omega_n$ i.e. equations (3). These are the *tangential Cauchy-Riemann equations* (on the boundary of the Siegel domain) and $\overline{L}_j$ are the *Lewy operators*. Indeed the operator $\overline{L}_1$ has been discovered by H. Lewy in his fundamental paper of 1958 (while investigating the boundary behavior of holomorphic functions on the Siegel domain in $\mathbb{C}^2$, cf. [20]).

The span of $\{L_j(p) : 1 \leq j \leq n\}$ at each $p \in \partial\Omega_n$ gives a complex rank n subbundle $T_{1,0}(\partial\Omega_n)$ of the complexified tangent bundle $T(\partial\Omega_n) \otimes \mathbb{C}$, the *CR structure* of $\partial\Omega_n$ satisfying

$$T_{1,0}(M)_p \cap T_{0,1}(M)_p = (0), \quad p \in M \equiv \partial\Omega_n\,, \tag{4}$$

$$Z, W \in C^\infty(U, T_{1,0}(M)) \Longrightarrow [Z, W] \in C^\infty(U, T_{1,0}(M)), \tag{5}$$

where $T_{0,1}(\partial\Omega_n) = \overline{T_{1,0}(\partial\Omega_n)}$ and $U \subset \partial\Omega_n$ is an open set. According to the calculations above, the CR structure $T_{1,0}(\partial\Omega_n)$ is *induced* by the complex structure of the ambient space $\mathbb{C}^{n+1}$ in the sense that

$$T_{1,0}(\partial\Omega_n)_p = [T_p(\partial\Omega_n) \otimes_{\mathbb{R}} \mathbb{C}] \cap T^{1,0}\left(\mathbb{C}^{n+1}\right)_p\,, \quad p \in \partial\Omega_n,$$

where $T^{1,0}(\mathbb{C}^{n+1})$ is the holomorphic tangent bundle over $\mathbb{C}^{n+1}$. Clearly these considerations admit a *verbatim* repetition to make sense of a notion of induced CR structure on the boundary of any domain $\Omega \subset \mathbb{C}^{n+1}$ (whose boundary is smooth), and actually on any smooth real hypersurface $M \subset \mathbb{C}^{n+1}$.

Let $M \subset \mathbb{C}^{n+1}$ be a real hypersurface, endowed with the induced CR structure $T_{1,0}(M)$, and let us consider the first order differential operator

$$\overline{\partial}_b : C^\infty(M, \mathbb{C}) \to C^\infty\left(T_{0,1}(M)^*\right),$$
$$(\overline{\partial}_b f)\overline{L} = \overline{L}(f), \quad f \in C^\infty(M, \mathbb{C}), \quad L \in T_{1,0}(M).$$

This is the *tangential Cauchy-Riemann operator* and $\overline{\partial}_b f = 0$ are the *tangential Cauchy-Rimann equations* on M (clearly equivalent to (3) when M is the boundary of the Siegel domain). A solution to $\overline{\partial}_b f = 0$ is a *Cauchy-Riemann function.* We shall not worry at this time for the regularity to be requested *a priori*, or to be expected *a posteriori*, for such a CR function. Note that for any holomorphic function $F \in \mathcal{O}(U)$, defined on an open set $U \subset \mathbb{C}^{n+1}$ such that $U \cap M \neq \emptyset$, the trace $f = F|_{U \cap M}$ is a CR function on $U \cap M$. We assume tacitly that M is embedded, so that $U \cap M$ be open in M. The converse is one of the main problems in CR geometry (the *CR extension problem*) and clearly admits a local, as well as a global, formulation.

Let $(M, T_{1,0}(M))$ be an *abstract* CR manifold i.e. a real $(2n+1)$-dimensional C^∞ manifold endowed with the a complex rank n subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ obeying to (4)-(5). Another fundamental problem in CR geometry (the *embedding problem*) is whether an embedding of M in $\mathbb{C}^{n+1}$ exists such that $T_{1,0}(M)$ be the CR structure induced by the complex structure of $\mathbb{C}^{n+1}$. The embedding problem clearly admits a local version as well and is still related to the tangential Cauchy-Riemann equations, for given an embedding $f = (f^1, \cdots, f^{n+1}) : M \to \mathbb{C}^{n+1}$ such that

$$(d_p f) T_{1,0}(M)_p = [T_p(M) \otimes_{\mathbb{R}}] \cap T\left(\mathbb{C}^{n+1}\right)_{f(p)}, \quad p \in M,$$

it may be easily seen that each f^j is a CR function on M. Hence, to solve the embedding problem one needs to produce enough functionally independent solutions to $\overline{\partial}_b f = 0$.

Let us go back to the boundary of the Siegel domain $\Omega_1 \subset \mathbb{C}^2$. Precisely, let $\mathbb{H} = \mathbb{C} \times \mathbb{R}$ be the Heisenberg group i.e. the 3-dimensional Lie group with the law

$$(z,t) \cdot (w,s) = (z + w,\ t + s + 2\,\mathrm{Im}(z\,\overline{w})), \quad (z,t),\, (w,s) \in \mathbb{H}.$$

The map

$$f : \mathbb{H} \to \partial\Omega_1, \quad f(z,t) = (z,\ t + i\,|z|^2), \quad (z,t) \in \mathbb{H},$$

is a C^∞ diffeomorphism of $\mathbb{H}$ onto the boundary of the Siegel domain and f^{-1} maps the Lewy operator $\overline{L}_1$ into $\overline{L}$ where

$$L = \frac{\partial}{\partial z} + i\,\overline{z}\,\frac{\partial}{\partial t}\,.$$

The span of L over $\mathbb{C}$ is a CR structure $T_{1,0}(\mathbb{H})$ and then f is a *CR isomorphism* i.e. a diffeomorphism such that

$$(d_p f) T_{1,0}(\mathbb{H}) = T_{1,0}(\partial\Omega_1)_{f(p)}\,, \quad p \in \mathbb{H}.$$

Lewy's operator L possesses well known unsolvability properties i.e. there is a smooth f such that $\overline{L}u = f$ has no smooth solution in any neighborhood of the origin, eliminating a belief (popular at the beginning of the 1950s) that linear first order equations with smooth coefficients should always have solutions (of course nonlinear examples trivially exist e.g. $e^{u'} = 0$ has no solutions). Solvability turns to be closely related to (local) embeddability as shown by Hill's example (cf. [14]). The equation $\overline{L}u = \omega$ (with $\omega \in C^\infty(\mathbb{H}, \mathbb{C})$ is *solvable* at $(z_0, t_0) \in \mathbb{R}^3$ if there is an open set $U \subset \mathbb{R}^3$ and a function $u \in C^\infty(U, \mathbb{C})$ such that $\overline{L}u = \omega$ in U. Let $T_{1,0}(\mathbb{H} \times \mathbb{C})$ be the CR structure spanned by

$$P = \frac{\partial}{\partial\overline{\zeta}}\,, \quad Q = \overline{L} + \omega(z,t)\,\frac{\partial}{\partial\zeta}\,,$$

where $\zeta : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is the projection. The result of C.D. Hill alluded to above is that $T_{1,0}(\mathbb{H} \times \mathbb{C})$ is locally embeddable at (z_0, t_0, ζ_0) if and only if $\overline{L}u = \omega$ is solvable at (z_0, t_0).

Let $\Psi : M \to \mathbb{C}^n$ be a C^1-embedding of a real m-dimensional manifold M. A point $x \in M$ is a *complex tangent* of $\Psi(M)$ if

$$[(d_x\Psi)T_x(M)] \cap \left[J_{\Psi(x)}(d_x\Psi)T_x(M)\right] \neq (0)$$

where J is the complex structure on $\mathbb{C}^n$. Let $\aleph_{\Psi(M)}$ be the set of all complex tangents of $\Psi(M)$. For each $x \in \aleph_{\Psi(M)}$ let $H(M)_x$ be the maximal complex subspace of $T_x(M)$ i.e.

$$(d_x\Psi)H(M)_x = [(d_x\Psi)T_x(M)] \cap \left[J_{\Psi(x)}(d_x\Psi)T_x(M)\right].$$

The dimension $\dim_{\mathbb{C}} H(M)_x$ is the *degree* of x. When $\Psi(M)$ is a real hypersurface in $\mathbb{C}^n$ one has $\aleph_{\Psi(M)} = M$ and each complex tangent $x \in M$ has degree $n-1$. In general $\aleph_{\Psi(M)}$ has a rather complicated topological structure (e.g. it is singular, stratified, etc.).

An embedding $\Psi : M \to \mathbb{C}^n$ is *totally real* if $\aleph_{\Psi(M)} = \emptyset$ and if this is the case it must be that $m \leq n$, and the case $m = n$ is of particular interest. Not all real n-dimensional manifolds admit totally real embeddings in $\mathbb{C}^n$. For instance, by a result of M. Gromov the only spheres S^n admitting a totally real embedding $S^n \to \mathbb{C}^n$ are S^1 and S^3. Ahern and Rudin exhibited an explicit totally real embedding $S^3 \to \mathbb{C}^3$ (cf. [1]).

A *knot* is a continuous simple embedding of S^1. Two knots $K_i \subset S^3$ $(i = 1, 2)$ are *topologically equivalent*, or have the same *topological type*, if there is a homeomorphism $h : S^3 \to S^3$ such that $h(K_1) = K_2$. A beautiful result by A.L. Elgindi (cf. [8]) is that for every knot $K \subset S^3$ and every positive integer $n \in \mathbb{N}$ there is an embedding $\Psi : S^3 \to \mathbb{C}^3$ of class C^n such that $\aleph_{\Psi(S^3)}$ is a knot topologically equivalent to K. That is every topological type of a knot in S^3 arises as the set of all complex tangents to some embedding of S^3 into $\mathbb{C}^3$. Remarkably the proof of Elgindi's result relies on the following facts

i) If $M = \rho^{-1}(0) \subset \mathbb{C}^2$ is a real hypersurface and $\Psi : M \to \mathbb{C}^3$ the embedding of M in $\mathbb{C}^3$ as the graph of a given smooth function $f : M \to \mathbb{C}$

$$\Psi(z, w) = (z, w, f(z, w)), \quad (z, w) \in M,$$

then

$$\aleph_{\Psi(M)} = \left(\overline{L}f\right)^{-1}(0).$$

ii) The map $\overline{L} : \mathbb{C}[z, w, \overline{z}, \overline{w}] \to \mathbb{C}[z, w, \overline{z}, \overline{w}]$ is (linear and) surjective.

A moral conclusion is that a deep circle of ideas relates the notions above and that a research project titled say *Complex tangents, knots and analytic discs, solvability and embeddability, through the geometry of canonical bundles* would be as actual today as well as at the time of H. Jacobowitz's results on this matter (cf. [15]).

2. Hörmander systems and the Jost-Xu program

Let

$$L = \frac{1}{2}(X - iY),$$

$$X = \frac{\partial}{\partial x} + 2y\,\frac{\partial}{\partial t}\,, \quad Y = \frac{\partial}{\partial y} - 2x\,\frac{\partial}{\partial t}\,,$$

be the real and imaginary parts of L. The vector fields $\{X,\ Y,\ \partial/\partial t\}$ are left invariant and form a basis of the Lie algebra of $\mathbb{H}$. Note that

$$[X, Y] = -4\,\frac{\partial}{\partial t}$$

(*Heisenberg's commutation* relation) hence the vector fields $\{X, Y\}$ and their commutator span the tangent space to $\mathbb{H} = \mathbb{R}^3$ at each point. Hence $\{X, Y\}$ is a special instance of what one commonly calls a Hörmander system on $\mathbb{R}^3$. Let $\{X^*,\ Y^*\}$ be the formal adjoints of $\{X, Y\}$ e.g.

$$\int_{\mathbb{H}} X^* f\ \varphi\ d\mu = \int_{\mathbb{H}} f\ X\varphi\ d\mu$$

for all $f, \varphi \in C^1_0(\mathbb{H})$ (actually $X^* = -X$ and $Y^* = -Y$ yet irrelevant). Next let us consider the second order differential operator

$$\Delta_b u = X^* X u + Y^* Y u$$

(commonly referred to as a Hörmander *sum of squares*). Let $\Omega \subset \mathbb{H}$ be a bounded domain with smooth boundary. The space $\mathrm{DO}_2(\overline{\Omega})$ of all second order differential operators

$$A = \sum_{|\alpha| \le 2} A_\alpha(x) D^\alpha$$

with real valued continuous coefficients on $\overline{\Omega}$, is a Banach space with the norm

$$\|A\| = \sum_{|\alpha| \le 2} \sup_{x \in \overline{\Omega}} |A_\alpha(x)|\,.$$

If $\mathrm{EO}_2(\overline{\Omega})$ is the subset consisting of all second order differential operators which are elliptic in $\overline{\Omega}$ then $\Delta_b \in \partial\,\mathrm{EO}_2(\overline{\Omega})$ i.e. Δ_b is a boundary point of $\mathrm{EO}_2(\overline{\Omega})$ and Δ_b is a *degenerate elliptic* operator in this sense. More explicitly, a calculation of the symbol of Δ_b shows that ellipticity degenerates precisely in the cotangent directions $\theta_{0,p}$, $p \in \mathbb{H}$, where

$$\theta_0 = dt + i \sum_{j=1}^{n} \left\{ z^j\, d\overline{z}_j - \overline{z}_j\, dz^j \right\}. \tag{6}$$

However Δ_b is *subelliptic* of order $\epsilon = 1/2$ i.e. for every point $p \in \mathbb{H}$ there is an open neighborhood $U \subset \mathbb{H}$ of p such that

$$\|u\|_{1/2}^2 \leq C \left(|(\Delta_b u\, ,\, u)_{L^2}| + \|u\|_{L^2}^2 \right), \quad u \in C_0^\infty(U),$$

where $\| \cdot \|_\epsilon$ is the Sobolev norm of order ϵ

$$\|u\|_\epsilon = \left(\int_{\mathbb{R}^3} \left(1 + |\xi|^2\right)^\epsilon \, |\hat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

($\hat{u}$ is the Fourier transform of u). Then, by a result of J.J. Kohn, Δ_b is hypoelliptic i.e. if $\Delta_b u = f$ in distributional sense and $f \in C^\infty$ then $u \in C^\infty$ as well. Hypoellipticity is the main property that Δ_b shares with elliptic operators.

Let N be a Riemannian manifold, carrying the Riemannian metric h, which may be covered by a single coordinate neighborhood $\chi = (y^1, \cdots, y^m) : N \to \mathbb{R}^m$, and let us consider the energy functional

$$E_\Omega : C^\infty(\mathbb{H}, N) \to \mathbb{R},$$

$$E_\Omega(\phi) = \frac{1}{2} \int_\Omega \sum_{a=1}^{2} X_a(\phi^j) X_a(\phi^k) \left(h_{jk} \circ \phi\right) \, d\mu \,,$$

where $X_1 = X$ and $X_1 = Y$ and μ is the Lebesgue measure on $\mathbb{R}^3$. Also

$$\phi^j = y^j \circ \phi, \quad h_{jk} = h(\partial/\partial y^j \, , \, \partial/\partial y^k), \quad 1 \leq j, k \leq m.$$

The Euler-Lagrange equations of the variational principle $\delta \, E_\Omega(\phi) = 0$ are

$$(H_N \phi)^i \equiv -\Delta_b \phi^i + \sum_{a=1}^{2} X_a(\phi^j) X_a(\phi^k) \, \left(\Gamma^i_{jk} \circ \phi\right) = 0 \tag{7}$$

and a solution $\phi \in C^\infty(\mathbb{H}, N)$ to (7) is a *subelliptic harmonic* map. The notion is due to J. Jost & C-J. Xu, who started a program of recovering results known for quasilinear elliptic systems of variational origin, such as the harmonic map system, to the at least hypoelliptic case. Clearly (7) can be made sense of for an arbitrary Hörmader system of vector fields, on an open subset $U \subset \mathbb{R}^N$, and if these vector fields are the coordinate vector fields $\{\partial/\partial x^A : 1 \leq A \leq N\}$ a subelliptic harmonic map is nothing but a harmonic map $\phi : U \to N$. Also a notion of *subelliptic harmonic morphism* is ready available for one may consider continuous maps $\phi : \mathbb{H} \to N$ pulling back local harmonics on N into local harmonics of Δ_b i.e.

$$\forall \;\; v : V \subset N \to \mathbb{R}, \quad \Delta_h v = 0 \;\;\text{ in }\;\; V \Longrightarrow \Delta_b(v \circ \phi) = 0 \;\;\text{ in }\;\; \phi^{-1}(V).$$

Here Δ_h is the ordinary Laplace-Beltrami operator of (N,h) on functions. Just as in the classical case of harmonic morphisms among Riemannian manifolds $\phi : \mathbb{H} \to N$ must be smooth. Indeed one may choose, about each point of N, a harmonic local coordinate system (V, y^j) and then $\phi^j = y^j \circ \phi$ must be harmonics of Δ_b i.e. $\Delta_b \phi^i = 0$. Yet Δ_b is hypoelliptic hence each ϕ^i, and then ϕ, is C^∞.

So Jost & Xu's subeliptic harmonic maps are no doubt a generalization of harmonic maps, and just as S. Campanato, M. Giaquinta, S. Hildebrand, H. Kaul, and K. Widman at their time, Jost & Xu may solve the Dirichlet problem for maps with values in regular balls, exhibiting a wonderful use of the so called *subelliptic technicalities* born with the work by E. Lanconelli and considerably developed into an autonomous and respectable chapter of PDEs theory by a number of people, among which the most important is perhaps N. Garofalo. Jost & Xu's result (cf. [16]) may be loosely stated as follows.

Theorem 1. *Let $\Omega \subset \mathbb{H}$ be a bounded domain with smooth boundary. Let (N,h) be a complete Riemannian manifold, covered by a single coordinate neighborhood. Let us assume the sectional curvature of (N,h) is bounded by above by a constant κ^2 and let $p \in N$ be a point and $0 < \mu < \min\{\pi/(2\kappa)\,,\, i(p)\}$ where $i(p)$ is the injectivity radius of p. Let $f \in C\left(\overline{\Omega}, N\right) \cap W^{1,2}_{X,Y}(\Omega, N)$ such that*

$$f\left(\overline{\Omega}\right) \subset B(p,\mu) = \{q \in N : d_h(p,q) < \mu\}.$$

Then there is a unique map $\phi \in W^{1,2}_{X,Y}(\Omega, N) \cap L^\infty(\Omega, N)$ such that $\phi|_{\partial\Omega} = f$ and $\phi(\overline{\Omega}) \subset B(p,\mu)$ and ϕ minimizes E_Ω among such maps. Also this ϕ is a weak solution to $H_N \phi = 0$. Moreover ϕ has the same interior regularity properties as solutions to linear hypoelliptic systems and if $\partial\Omega$ is C^∞ and noncharacteristic for $\{X,Y\}$ and if f is C^∞ one gets the corresponding boundary regularity of ϕ.

Here $W^{1,2}_{X,Y}(\Omega, N)$ are Sobolev-type spaces of maps whose components admit weak L^2 derivatives in the directions X and Y. The boundary $\partial\Omega$ is *noncharacteristic* for the system $\{X,Y\}$ if for every point $p \in \partial\Omega$ either $X_p \notin T_p(\partial\Omega)$ or $Y_p \notin T_p(\partial\Omega)$. It may be easily shown (e.g. by looking at generalized solutions to $\Delta_b u = f$ i.e. solutions $u \in C^1$ whose second order derivatives have jumps across $\partial\Omega$) that the (non) characteristic notion just introduced is the ordinary notion in the theory of characteristics and of the Cauchy problem for $\Delta_b u = f$.

Our point of view in this talk is that Jost & Xu's notion (of a subelliptic harmonic map) is rather formal[1] (in the end just replacing by Δ_b the ordinary Laplacian in the harmonic map system) and wish to exhibit a geometric interpretation.

The differential 1-form θ_0 given by (6) above is a *contact form* on $\mathbb{H}$ i.e. $\theta_0 \wedge d\theta_0$ is a volume form on $\mathbb{H}$. It is but one of the infinitely many nowhere zero sections of the conormal bundle $H(\mathbb{H})^\perp \subset T^*(\mathbb{H})$ where $H(\mathbb{H})$ is the *Levi distribution* of $\mathbb{H}$ as a CR manifold, with the CR structure $T_{1,0}(\mathbb{H})$. Precisely

$$H(\mathbb{H}) = \mathrm{Re}\left\{T_{1,0}(\mathbb{H}) \cap T_{0,1}(\mathbb{H})\right\},$$
$$H(\mathbb{H})_p^\perp = \{\omega \in T_p^*(\mathbb{H}) : \mathrm{Ker}(\omega) \supset H(\mathbb{H})_p\}, \quad p \in \mathbb{H}.$$

Then $H(\mathbb{H})^\perp \to \mathbb{H}$ is a real line bundle (as $\mathbb{H}$ is oriented and connected this bundle is actually trivial i.e. $H(\mathbb{H})^\perp \approx \mathbb{H} \times \mathbb{R}$) and any other nowhere zero section $\theta \in C^\infty(H(\mathbb{H})^\perp)$ (any *pseudohermitian structure* on $\mathbb{H}$, according to the terminology introduced by S. Webster) is "conformally" related to θ_0 i.e. $\theta = \lambda\,\theta_0$ for some C^∞ function $\lambda : \mathbb{H} \to \mathbb{R} \setminus \{0\}$. In particular any pseudohermitian structure on $\mathbb{H}$ is a contact form. The Levi distribution carries the complex structure

$$J : H(\mathbb{H}) \to H(\mathbb{H}),$$
$$J(Z + \overline{Z}) = i(Z - \overline{Z}), \quad Z \in T_{1,0}(\mathbb{H}).$$

Given any contact form $\theta \in C^\infty(H(\mathbb{H})^\perp)$ the *Levi form* is

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(\mathbb{H}).$$

As an immediate consequence of definitions

$$G_\theta = \lambda\, G_{\theta_0}$$

accounting for the (largely exploited) analogy among CR and conformal geometry. An easy verification shows that G_{θ_0} is positive definite hence $(\mathbb{H},\, T_{1,0}(\mathbb{H}))$ is *strictly pseudoconvex*, a notion which is immediately

[1]Although the functional $E_\Omega : C^\infty(\mathbb{H}, N) \to \mathbb{R}$ above isn't postulated but rather *discovered* by setting

$$X_j = \sum_{A=1}^{3} b_j^A(x)\,\frac{\partial}{\partial x^A}\,, \quad a^{AB}(x) = \sum_{j=1}^{2} b_j^A(x) b_j^B(x),$$

$$(g_\epsilon)^{AB} = a^{AB} + \epsilon\,\delta^{AB}\,, \quad [(g_\epsilon)_{AB}] = \left[(g_\epsilon)^{AB}\right]^{-1}\,, \quad \epsilon > 0,$$

$$E_{\Omega,\epsilon}(\phi) = \frac{1}{2}\int_\Omega \|d\phi\|^2_{g_\epsilon}\, dx = \frac{1}{2}\int_\Omega (g_\epsilon)^{AB}\,\frac{\partial \phi^\alpha}{\partial x^A}\,\frac{\partial \phi^\beta}{\partial x^B}\,(h_{\alpha\beta}\circ\phi)\; dx$$

(the ordinary Dirichlet functional on smooth maps among the Riemannian manifolds (M, g_ϵ) with $g_\epsilon(\partial_A\,,\, \partial_B) = (g_\epsilon)_{AB}$ and (N, h)) and letting $\epsilon \to 0$ yields $E_{\Omega,\epsilon}(\phi) \to E_\Omega(\phi)$.

made sense of on an abstract CR manifold. If θ is chosen such that G_θ be positive definite (i.e. $\lambda > 0$) then one may consider the Riemannian metric g_θ (the *Webster metric*) given by

$$g_\theta(X,Y) = G_\theta(X,Y), \quad g_\theta(X,T) = 0, \quad g_\theta(T,T) = 1,$$

for any $X, Y \in H(\mathbb{H})$. Here $T \in \mathfrak{X}(\mathbb{H})$ is the *Reeb vector* of $(\mathbb{H}, \theta)$ i.e. the tangent vector field on $\mathbb{H}$ determined by

$$\theta(T) = 1, \quad T \rfloor d\theta = 0.$$

Then $(H(\mathbb{H}), G_\theta)$ is a sub-Riemannian structure on $\mathbb{H}$ and the Webster metric g_θ *contracts* the sub-Riemannian structure i.e.

$$d_{g_\theta}(p,q) \leq d_H(p,q), \quad p, q \in \mathbb{H},$$

where d_{g_θ} and d_H are respectively the distance function associated to the Riemannian structure g_θ and the Carnot-Carthéodory distance function. A rich geometric structure arises from the CR structure $T_{1,0}(\mathbb{H})$ alone. To this we shall add in a moment geometric objects such as the Tanaka-Webster connection, and the Fefferman metric. It is an accepted philosophy that the study of these geometric objects may ultimately shed light on the properties of the solutions to the tangential Cauchy-Riemann equations. We shall use these objects, and others to come, such as the Graham-Lee connection, in order to give the geometric interpretation of subelliptic harmonic maps alluded to above.

It should be mentioned that, in a couple of papers published at the end of the 1970's in Trans. A.M.S and Proc. A.M.S. (cf. [3]) A Bejancu also introduced a notion of *CR submanifold M* of a given Kählerian manifold $\tilde{M}$, as a real submanifold carrying a distribution $\mathcal{D}$ such that i) $\tilde{J}\mathcal{D} \subset \mathcal{D}$ and ii) the orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$ in $(T(\tilde{M}), \tilde{g})$ satisfies $\tilde{J}\mathcal{D}^\perp \subset T(M)^\perp$. Here $\tilde{J}$ and $\tilde{g}$ are respectively the complex structure and the Kählerian metric on $\tilde{M}$. The notion was soon embraced by the portion of mathematical community devoted to the study of the geometry of second fundamental form of isometric immersions among Riemannian manifolds, a large amount of papers on CR submanifolds of Kählerian manifolds first and then of Hermitian manifolds of sorts (such as locally conformal Kähler manifolds) have been published, and eventually A. Bejancu got the Romanian Academy price for his discovery. An observation by D.E. Blair and B-Y. Chen shows that, as a consequence of the Kähler condition on $\tilde{g}$

$$\{X - i\tilde{J}X \in T(M) \otimes \mathbb{C} : X \in \mathcal{D}\}$$

is a CR structure on M, though of higher CR codimension (actually a locally conformal Kähler metric $\tilde{g}$ suffices). So it seems a circle has been closed, and A. Bejancu's notion does fit into CR geometry, in spite of some criticism formulated at the time of its discovery (both because Blair and Chen's observation appeared somewhat later, and because A. Bejancu's CR submanifolds were already embedded, while it appears he was unaware of when CR manifolds embed e.g. unaware of Andreotti and Hill's embedability theorem). It should be however observed that the study of the geometry of the second fundamental form of a CR submanifold of a Hermitian manifold, based as it is on the use of the first fundamental form and of Gauss-Weingarten and Gauss-Codazzi-Ricci equations, is confined to Riemannian geometry: if for instance $\tilde{M} = \mathbb{C}^n$ and M is the boundary of the Siegel domain then *none* of the infinitely many Webster metrics of M coincides with the first fundamental form (the metric induced on M by the flat Kähler metric of the ambient space $\mathbb{C}^n$). In turn the Webster metrics $\{g_\theta : \theta \in \mathcal{P}\}$ of a (at least nondegenerate) CR manifold and their curvature describe the pseudoconvexity properties of M, as understood in complex analysis in several complex variables.

3. Fefferman's metric

Let $(M, T_{1,0}(M))$ be a CR manifold, of real dimension $2n+1$ and CR dimension n. Let $H(M)$ be its Levi distribution. Assume that M is strictly pseudoconvex i.e. the Levi form G_θ is positive definite for some pseudohermitian structure θ on M. Let $T \in \mathfrak{X}(M)$ be the Reeb vector of (M, θ) and g_θ the Webster metric. By a result got independently by S. Webster and N. Tanaka, there is a unique linear connection ∇ on M such that i) $H(M)$ is parallel with respect to ∇, ii) $\nabla J = 0$, $\nabla g_\theta = 0$, and iii) the torsion tensor field T_∇ of ∇ is *pure* i.e.

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2i G_\theta(Z, \overline{W}), \quad Z, W \in T_{1,0}(M),$$

$$\tau \circ J + J \circ \tau = 0, \quad \tau(X) \equiv T_\nabla(T, X), \quad X \in \mathfrak{X}(M).$$

∇ is the *Tanaka-Webster connection* of (M, θ) (cf. [21], [22]). The assumption of nondegeneracy of some Levi form G_θ (and thus of all) actually suffices to prove existence and uniqueness of such ∇ (we shall only exploit the result in the strictly pseudoconvex cases).

A complex valued differential p-form ω on M is a $(p,0)$-*form* if $T_{0,1}(M) \rfloor \omega = 0$. Let $\Lambda^{p,0}(M) \to M$ be the relevant bundle. Then $K(M) = \Lambda^{n+1,0}(M)$ is a complex line bundle over M (the *canonical line* bundle). Let us set

$$C(M) = \left[K(M) \setminus \{\text{zero section}\}\right] / \mathbb{R}^+, \quad \mathbb{R}^+ = \mathrm{GL}^+(1, \mathbb{R}),$$

so that to obtain a principal S^1-bundle

$$S^1 \to C(M) \xrightarrow{\pi} M$$

(the *canonical circle bundle*). Let $F_\theta \in \mathrm{Lor}(C(M))$ be the Lorentzian metric (the *Fefferman metric* of (M, θ)) given by

$$F_\theta = \pi^* \tilde{G}_\theta + 2\,(\pi^*\theta) \odot \sigma,$$

$$\sigma = \frac{1}{n+2} \left\{ d\gamma + \pi^* \left[i\,\omega_\alpha{}^\alpha - i\, g^{\alpha\overline{\beta}}\, dg_{\alpha\overline{\beta}} + \frac{\rho}{4(n+1)}\, \theta \right] \right\},$$

$$\tilde{G}_\theta(X,Y) = G_\theta(X,Y), \quad \tilde{G}_\theta(T,\,\cdot\,) = 0, \quad X, Y \in H(M),$$

$$\nabla T_\beta = \omega_\beta{}^\alpha T_\alpha\,, \quad \rho = g^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}\,,$$

$$g_{\alpha\overline{\beta}} = G_\theta(T_\alpha, T_{\overline{\beta}}), \quad \left[g^{\alpha\overline{\beta}} \right] = \left[g_{\alpha\overline{\beta}} \right]^{-1}\,,$$

$$R_{\alpha\overline{\beta}} = \mathrm{Ric}_\nabla(T_\alpha, T_{\overline{\beta}}), \quad \mathrm{Ric}_\nabla(X,Y) = \mathrm{trace}\left\{ Z \longmapsto R^\nabla(Z,Y)X \right\},$$

for any local frame $\{T_\alpha : 1 \leq \alpha \leq n\} \subset C^\infty(U, T_{1,0}(M))$. Also γ is a local fibre coordinate on $C(M)$. By a result of C.R. Graham, σ is a connection 1-form in the principal bundle $S^1 \to C(M) \xrightarrow{\pi} M$. By a result of J.M. Lee

$$\hat{\theta} = e^u \theta \Longrightarrow F_{\hat{\theta}} = e^{u \circ \pi} F_\theta$$

and then $\{e^{u\circ\pi} F_\theta : u \in C^\infty(M)\}$ (the *restricted conformal class* of F_θ) is a CR invariant. None of the Fefferman metrics F_θ is Einstein, yet $(C(M), F_\theta)$ is a space-time with the time orientation $T^\uparrow - S$ (here S is the tangent to the action of S^1 on $C(M)$ and $T^\uparrow$ is the horizontal lift of T with respect to the connection 1-form σ).

Let us assume for simplicity that M is compact (hence $C(M)$ is compact) and consider the energy functional

$$\mathbb{E}(\Phi) = \frac{1}{2} \int_{C(M)} \mathrm{trace}_{F_\theta}\left(\Phi^* h\right)\; d\mathrm{vol}(F_\theta),$$

$$\Phi \in C^\infty(C(M), N),$$

where (N, h) is a Riemannian manifold. If F_θ were a Riemannian metric, the trace $\mathrm{trace}_{F_\theta}\left(\Phi^* h\right)$ would be the (squared) Hilbert-Schmidt norm $\|d\Phi\|^2$. While harmonic maps theory arises in the Riemannian category, it is long consolidated within the semi-Riemannian category, as well (starting with the work by B. Fuglede). A C^∞ map $\Phi : C(M) \to N$ is *harmonic* if

$$\frac{d}{dt}\left\{\mathbb{E}(\Phi_t)\right\}_{t=0} = 0$$

for any smooth 1-parameter variation $\{\Phi_t\}_{|t|<\epsilon} \subset C^\infty(C(M), N)$ of Φ i.e. $\Phi_0 = \Phi$. The Euler-Lagrange equations of $\delta\,\mathbb{E}(\Phi) = 0$ are

$$-\Box\Phi^i + \left(\Gamma^i_{jk} \circ \Phi\right) \frac{\partial \Phi^j}{\partial u^A} \frac{\partial \Phi^k}{\partial u^B} (F_\theta)^{AB} = 0 \tag{8}$$

where $\Box$ is the Laplace-Beltrami operator of $(C(M), F_\theta)$ (the *wave operator*) and $u^0 = \gamma$, $u^A = x^A \circ \pi$, $1 \le A \le 2n+1$, are local coordinates on $C(M)$ (induced by the local coordinate system $(U, x^1, \cdots, x^{2n+1})$ on M). If $\Phi : C(M) \to N$ is S^1-invariant and $\phi : M \to N$ is the corresponding base map (so that $\Phi = \phi \circ \pi$) then one may integrate over the fibres in $\mathbb{E}(\phi \circ \pi)$ so that to get[2]

$$\mathbb{E}(\phi \circ \pi) = \pi \int_M \mathrm{trace}_{G_\theta} \left\{\Pi_H\left(\phi^* h\right)\right\}\ \theta \wedge (d\theta)^n \equiv 2\pi\, E(\phi)$$

where $\Pi_H(\phi^* h)$ is the restriction of $\phi^* h$ to $H(M) \otimes H(M)$. One has then discovered the energy integral which is appropriate for subelliptic theory, for $\phi \in C^\infty(M, N)$ is a critical point of E if and only if $\phi \circ \pi$ is a harmonic map of $(C(M), F_\theta)$ into (N, h) (by a result of E. Barletta et al.) and the Euler-Lagrange equations of $\delta\, E(\phi) = 0$ are

$$-\Delta_b \phi^i + \sum_{a=1}^{2n} X_a(\phi^j) X_a(\phi^k) \left(\Gamma^i_{jk} \circ \phi\right) = 0 \tag{9}$$

where $\{X_a : 1 \le a \le 2n\} \subset C^\infty(U, H(M))$ is a local G_θ-orthonormal frame. Equations (9) are actually the projection on M via π of the harmonic map system (8) since (by a result of J.M. Lee) $\pi_* \Box = \Delta_b$. Here Δ_b (the *sublaplacian* of (M, θ)) is the second order differential operator given by

$$\Delta_b u = -\mathrm{div}\left(\nabla^H u\right), \quad u \in C^2(M),$$

$$\mathcal{L}_X\ \left(\theta \wedge (d\theta)^n\right) = \mathrm{div}(X)\, \theta \wedge (d\theta)^n,$$

$$\nabla^H u = \Pi_H \nabla u\,, \quad \Pi_H : T(M) = H(M) \oplus \mathbb{R}T \to H(M),$$

$$g_\theta(\nabla u\,,\, X) = X(u), \quad X \in \mathfrak{X}(M).$$

On the other hand, if U is also the domain of a local chart (U, φ) on M, then it may be easily seen that $\{\varphi_* X_a : 1 \le a \le 2n\}$ is a Hörmander system on $\varphi(U) \subset \mathbb{R}^{2n+1}$ and $\Delta_b \equiv \sum_{a=1}^{2n} X_a^* X_a$, hence solutions to (9) are subelliptic harmonic maps (giving a first geometric interpretation to those).

It should be mentioned that Fefferman's metric was originally built in the following extrinsic manner. Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded

[2]The second π is the irrational number $\pi \in \mathbb{R} \setminus \mathbb{Z}$ (so does the third).

strictly pseudoconvex domain and let us consider the Dirichlet problem for the complex Monge-Ampére equation

$$J(u) \equiv (-1)^n \det \begin{pmatrix} u & \partial u/\partial \overline{z}_k \\ \partial u/\partial z_j & \partial^2 u/\partial z_j \, \partial \overline{z}_k \end{pmatrix} = 1 \quad \text{in } \Omega, \tag{10}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{11}$$

By a result of S-Y. Cheng & S-T. Yau the problem (10)-(11) admits a unique solution u which is C^∞ in the interior of Ω and belongs to $C^{n+(3/2)-\epsilon}(\overline{\Omega})$. Let us consider the function

$$H : \Omega \times (\mathbb{C} \setminus \{0\}) \to \mathbb{R},$$

$$H(z,\zeta) = |\zeta|^{2/(n+1)} \, u(z), \quad z \in \Omega, \quad \zeta \in \mathbb{C} \setminus \{0\},$$

where u is the unique solution to (10)-(11). Moreover we consider the $(0,2)$-tensor field G on $\Omega \times (\mathbb{C} \setminus \{0\})$ given by

$$G = \sum_{A,B=0}^{n} \frac{\partial^2 H}{\partial z^A \, \partial \overline{z}^B} \, dz^A \odot d\overline{z}^B$$

where $z^0 = \zeta$. Then G is a biholomorphic invariant of Ω, in the following sense. Let $F : \Omega \to \Omega$ be a biholomorphic map and let us set

$$\mathcal{F} : \Omega \times (\mathbb{C} \setminus \{0\}) \to \Omega \times (\mathbb{C} \setminus \{0\}) \, ,$$

$$\mathcal{F}(z,\zeta) = \left(F(z) \, , \, \frac{\zeta}{\det \, F'(z)} \right), \quad z \in \Omega, \;\; \zeta \in \mathbb{C} \setminus \{0\}.$$

Then (by a result of C. Fefferman) $\mathcal{F}$ is a biholomorphism of $\Omega \times (\mathbb{C} \setminus \{0\})$ in itself and $\mathcal{F}^* G = G$. One may also show that

$$\det \left(\frac{\partial^2 H}{\partial z^A \, \partial \overline{z}^B} \right) = \frac{J(u)}{(n+1)^2}$$

hence G is nondegenerate and actually a semi-Riemannian metric on $\Omega \times (\mathbb{C} \setminus \{0\})$. The explicit expression of G is

$$G = \frac{u(z)}{(n+1)^2} \, |\zeta|^{2/(n+1)-2} \, d\zeta \odot d\overline{\zeta} +$$

$$+ \frac{|\zeta|^{2/(n+1)}}{n+1} \, (\partial u) \odot (\frac{1}{\overline{\zeta}} \, d\overline{\zeta}) + \frac{|\zeta|^{2/(n+1)}}{n+1} \, (\frac{1}{\zeta} \, d\zeta) \odot (\overline{\partial} u) +$$

$$+ |\zeta|^{2/(n+1)} \, \frac{\partial^2 u}{\partial z^j \, \partial \overline{z}^k} \, dz^j \odot d\overline{z}^k \, .$$

Let Ω be given by the defining function φ i.e. $\Omega = \{\varphi < 0\}$ and $D\varphi(z) \neq 0$ for every $z \in \partial\Omega$. For each $\epsilon \geq 0$ let us set $M_\epsilon = \{z \in \Omega : \varphi(z) = -\epsilon\}$ (so that $M_0 = \partial\Omega$). Let $G_\epsilon = j_\epsilon^* G$ where $j_\epsilon : M_\epsilon \times S^1 \to$

$\Omega \times (\mathbb{C} \setminus \{0\})$ is the inclusion. It may be shown that G_ϵ tends, as $\epsilon \to 0$, to a Lorentzian metric g on $\partial\Omega \times S^1$. When $M = \partial\Omega$ and θ is a positively oriented contact form on M then $C(M)$ is trivial i.e. there is a principal bundle isomorphism $\Phi : C(M) \to M \times S^1$ and the Lorentzian metrics F_θ and g are conformally equivalent i.e. $\Phi^* g = e^f F_\theta$ for some $f \in C^\infty(C(M))$.

Fefferman's metric is both an object worth of further investigation and, within the limits of our present knowledge of its properties, a tool of first magnitude. By Fefferman's own work, if $M \subset \mathbb{C}^{n+1}$ is a strictly pseudoconvex real hypersurface then Chern-Moser's chains are the projections on M via $\pi : C(M) \to M$ of the non-vertical null geodesics of F_θ. This enabled L. Koch to give a new proof of the result by H. Jacobowitz, that any two nearby points on M may be joined by a chain. While Koch's proof of Jacobowitz's theorem isn't necessarily simpler, it has the advantage of showing that there is an alternative, and very appealing, approach to CR and pseudohermitian geometry through Lorenzian geometry.

A piece of folklore is that just any nice problem in CR geometry begins with, or is related to, Fefferman's metric. One illustrious instance of that is *CR Yamabe*'s problem i.e. given a contact form $\theta \in \mathcal{P}_+$ find $u \in C^\infty(M)$ such that (the Tanaka-Webster connection associated to the contact form) $e^u \theta$ has constant pseudohermitian scalar curvature: this turns out to be precisely the Yamabe problem for the Fefferman metric F_θ. Relevant equation is certainly non-elliptic but its projection on M turns out to be (nonlinear) subelliptic and then subelliptic technicalities (whose basics were developed for the occasion) may be used to prove existence of the solution. Fundamental work in this direction is due to D. Jerison and J.M. Lee, and to Gamara and Yacoub in small dimension. The geometric interpretation of Jost and Xu's subelliptic harmonic maps belongs here (it relies on projecting on M the harmonic maps system on $C(M)$).

By some of my older work, if $M = S^{2n+1}$ then the first Pontryagin form of F_θ vanishes (i.e. $P_1(\Omega^2) = 0$) and the corresponding transgression class is integral (i.e. $[TP_1(\omega)] \in H^3(L(C(M)), \mathbb{Z})$). In general Pontyagin forms of the Fefferman metric are CR invariants, and if some Pontryagin form vanishes then the corresponding transgression class is a CR invariant, as well. I understand there is a lot of progress in this area but, to my knowledge the characteristic rings of $C(M)$ and M have not been effectively related. Fefferman's metric (and corresponding Levi-Civita connection, which by the way may be very effectively related to the Tanaka-Webster connection) should be used to compute the Weil homomorphism $w : I(\mathrm{GL}(2n+2, \mathbb{R})) \to H^*(C(M), \mathbb{R})$.

4. Bergman kernel and metric

Let $\Omega \subset \mathbb{C}^n$ be a domain and $L^2(\Omega)$ the Lebesgue space of all measurable functions $f : \Omega \to \mathbb{C}$ such that

$$\|f\|_{L^2} = \left(\int_\Omega |f(z)|^2 \, d\mu(z) \right)^{1/2} < \infty$$

where μ is the Lebesgue measure on $\mathbb{R}^{2n}$. Let

$$(f, g)_{L^2} = \int_\Omega f(z) \, \overline{g(z)} \, d\mu(z)$$

be the inner product on $L^2(\Omega)$. Let $H^2(\Omega)$ be the space of all functions $f \in L^2(\Omega)$ which are holomorphic in Ω. Let $A \subset \Omega$ be a compact subset and let $f \in H^2(\Omega)$ an L^2 holomorphic function on Ω. Let $\zeta \in A$ and

$$P(\zeta, \rho) = \{ z \in \mathbb{C}^n : |z_j - \zeta_j| < \rho_j \, , \ \ 1 \le j \le n \}$$

a polidisc of poliradius $\rho = (\rho_1, \cdots, \rho_n)$ such that $\overline{P}(\zeta, \rho) \subset \Omega$ and f may be represented as a uniformly convergent power series

$$f(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha (z - \zeta)^\alpha \, , \quad z \in P(\zeta, \rho).$$

Then

$$\|f(z)\|_{L^2}^2 = \int_\Omega |f(z)|^2 \, d\mu(z) \ge \int_{\overline{P}(\zeta,\rho)} f(z) \, \overline{f(z)} \, d\mu(z) =$$

$$= \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} a_\alpha \, \overline{a}_\beta \int_{\overline{P}(\zeta,\rho)} (z - \zeta)^\alpha (\overline{z} - \overline{\zeta})^\beta \, d\mu(z) =$$

(as terms of the form $a_\alpha (z - \zeta)^\alpha$ and $a_\beta (z - \zeta)^\beta$ with $\alpha \neq \beta$ are mutually orthogonal with respect to the L^2 inner product $(g, h)_\rho = \int_{P(\zeta,\rho)} g(z) \overline{h(z)} \, d\mu(z)$)

$$= \sum_{|\alpha|=0}^{\infty} \int_{P(\zeta,\rho)} |a_\alpha|^2 \prod_{j=1}^{n} |z_j - \zeta_j|^{2\alpha_j} \ d\mu(z) \ge$$

$$\ge \left| a_{(0,\cdots,0)} \right|^2 \int_{P(\zeta,\rho)} d\mu(z) = |f(z)|^2 \, \mu \left(P(\zeta, \rho) \right)$$

that is

$$|f(z)| \le \mu \left(P(\zeta, \rho) \right)^{-1/2} \|f\|_{L^2} \, .$$

A compactness argument now shows that

$$|f(z)| \le C_A \|f\|_{L^2} \tag{12}$$

for some constant $C_A > 0$ (depending only on the compact set A) and any $f \in H^2(\Omega)$, $z \in A$. The simple estimate (12) plays a crucial role. First, it clearly means that for every $z \in \Omega$ the evaluation functional

$$\delta_z : H^2(\Omega) \to \mathbb{C}, \quad \delta_z(f) = f(z), \quad f \in H^2(\Omega),$$

is continuous. As a second, and a bit less obvious, consequence of (12) the subspace $H^2(\Omega)$ is closed in $L^2(\Omega)$ and hence a Hilbert space itself. By the Riesz representation theorem for every $z \in \Omega$ there is an L^2 holomorphic function $k_z \in H^2(\Omega)$ such that

$$\delta_z(f) = (f\ ,\ k_z)_{L^2}\ , \quad f \in H^2(\Omega),$$

or

$$f(z) = \int_\Omega K(z,\zeta)\, f(\zeta)\; d\mu(\zeta) \tag{13}$$

for any $f \in H^2(\Omega)$ and any $z \in \Omega$, where we have set $K(z,\zeta) = \overline{k_z(\zeta)}$ for any $z, \zeta \in \Omega$. The function $K : \Omega \times \Omega \to \mathbb{C}$ is the *Bergman kernel* of Ω (and identity (13) show that K *reproduces* the L^2 holomorphic functions).

If $\{\phi_\nu\}_{\nu \geq 0}$ is a complete orthonormal system in $H^2(\Omega)$ then

$$K(z,\zeta) = \sum_{\nu=0}^{\infty} \phi_\nu(z) \overline{\phi_\nu(\zeta)}$$

with uniform convergence on compact subsets $\Omega \times \Omega$. It follows that $K : \Omega \times \Omega \to \mathbb{C}$ is a C^∞ map, holomorphic in z and anti-holomorphic in ζ. Also if Ω is bounded than for each $z_0 \in \Omega$ there is $f \in H^2(\Omega)$ such that $f(z_0) \neq 0$ hence the identity

$$K(z,z) = \sum_{\nu=0}^{\infty} |\phi_\nu(z)|^2$$

implies that $K(z,z) > 0$ for every $z \in \Omega$. Consequently we may consider

$$g = \sum_{j,k=1}^{n} \frac{\partial^2 \log K(z,z)}{\partial z^j\, \partial \overline{z}^k}\, dz^j \odot d\overline{z}^k$$

which turns out to be a Kählerian metric on Ω (the *Bergman metric*) with respect to which biholomorphisms of Ω are isometries (i.e. $\mathrm{Hol}(\Omega) \subset \mathrm{Isom}(\Omega, g)$). Also if Ω is homogeneous (i.e. $\mathrm{Hol}(\Omega)$ is transitive on Ω) then (Ω, g) is Kähler-Einstein.

5. Fefferman's asymptotic expansion formula for the Bergman kernel and its consequences

If $\Omega = \mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}$ is the unit ball then its Bergman kernel is

$$K(z,\zeta) = \frac{C_n}{\left(1 - z \cdot \overline{\zeta}\right)^{n+1}}\,, \quad C_n = \frac{n!}{\pi^n}\,,$$

so its properties (e.g. smoothness at the boundary) can be read off directly from its expression. There are few other instances of domains $\Omega \subset \mathbb{C}^n$ where Bergman's kernel may be explicitly computed. However when Ω is a smoothly bounded strictly pseudoconvex domain then our knowledge rests on a few classical results as follows. One such result is N. Kerzman's theorem that $K \in C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \Delta)$ where $\Delta = \{(z,z) : z \in \partial\Omega\}$ is the diagonal of the boundary. Proof is a wonderful application of the solution to the $\overline{\partial}$-Neumann problem.

Another such result, with deep differential geometric implications as we shall see, is Fefferrman asymptotic formula for the Bergman kernel. Precisely, again for any smoothly bounded strictly pseudoconvex domain $\Omega = \{\rho < 0\} \subset \mathbb{C}^n$

$$K(\zeta, z) = C_\Omega |\nabla \rho(z)|^2 \cdot \det L_\rho(z) \cdot \Psi(\zeta, z)^{-(n+1)} + E(\zeta, z) \tag{14}$$

where $E \in C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \Delta)$ and

$$|E(\zeta, z)| \leq C'_\Omega |\Psi(\zeta, z)|^{-(n+1)+1/2} \cdot |\log|\Psi(\zeta, z)||. \tag{15}$$

Here L_ρ is the Levi form and

$$\Psi(\zeta, z) = [F(\zeta, z) - \rho(z)]\, \chi(|\zeta - z|) + (1 - \chi(|\zeta - z|))\, |\zeta - z|^2\,,$$

$$F(\zeta, z) = -\sum_{j=1}^{n} \frac{\partial \rho}{\partial z^j}(z)(\zeta^j - z^j) - \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z^j\, \partial z^k}(z)(\zeta^j - z^j)(\zeta^k - z^k),$$

and $\chi(t)$ is a C^∞ cut-off function with $\chi(t) = 1$ for $|t| < \epsilon_0/2$ and $\chi(t) = 0$ for $|t| \geq 3\epsilon_0/4$. By (14)

$$K(z,z)^{-1/(n+1)} = |\rho(z)| \left[\Phi(z) + E(z,z)\, |\rho(z)|^{n+1}\right]^{-1/(n+1)}$$

where $\Phi(z) \equiv C_\Omega |\nabla \rho(z)|^2 \det L_\rho(z)$ stays finite near $\partial\Omega$ and (by (15))

$$|E(z,z)|\, |\rho(z)|^{n+1} \leq C'_\Omega |\rho(z)|^{1/2} |\log|\rho(z)|| \to 0, \quad z \to \partial\Omega.$$

Therefore $K(z,z)^{-1/(n+1)}$ vanishes at $\partial\Omega$. Also, as $\Phi(z) \neq 0$ near $\partial\Omega$, it may be shown that $\nabla K(z,z)^{-1/(n+1)} \neq 0$ along $\partial\Omega$. Hence

$$\varphi(z) = -K(z,z)^{-1/(n+1)}$$

is a defining function for Ω. Let

$$\theta = \frac{i}{2}\left(\overline{\partial} - \partial\right)\varphi, \quad g = \frac{\partial^2 \log K(z,z)}{\partial z^j\, \partial \overline{z}^k}\, dz^j \odot d\overline{z}^k\,.$$

Mere differentiation shows that the Bergman metric g may be expressed as

$$g(X,Y) = \frac{n+1}{\varphi}\left\{\frac{i}{\varphi}\left(\partial\varphi \wedge \overline{\partial}\varphi\right)(X, JY) - (d\theta)(X, JY)\right\} \tag{16}$$

for any $X, Y \in \mathfrak{X}(\Omega)$, where J is the complex structure on $\mathbb{C}^n$. Formula (16) relates computationally the Käbler geometry of (Ω, g) to the contact geometry of $(\partial\Omega, \theta)$ and this relationship has already been exploited by A. Korányi and H.M. Reimann in a beautiful paper (cf. [17]) where they relax the hypothesis of Fefferman's theorem that biholomorphisms of smoothly bounded strictly pseudoconvex domains extend smoothly at the boundary. If $\Phi : \Omega \to \Omega$ is a boholomorphism then its boundary values $\phi : \partial\Omega \to \partial\Omega$ must be a CR isomorphism and in particular a contact transformation (i.e. $\phi_* H(\partial\Omega) \subset \phi^{-1} H(\partial\Omega)$). When $\Phi : \Omega \to \Omega$ is but a symplectomorphism (with respect to the symplectic structure $-i\,\partial\overline{\partial}\log K(z,z)$ underlying the Kählerian structure of (Ω, g)) extending smoothly at the boundary, the result by Korányi and Reimann alluded to is that its boundary values $\phi : \partial\Omega \to \partial\Omega$ are at least a contact transformation.

6. Boundary values of Bergman-harmonic maps

Let

$$M_\delta = \{z \in \Omega : \varphi(z) = -\delta\}, \quad \delta > 0,$$

be the level sets of φ. For δ sufficiently small M_δ is still a smooth real hypersurface, and a strictly pseudoconvex CR manifold for that matter. Therefore there is a one-sided neighborhood $V \subset \overline{\Omega}$ of $\partial\Omega$ carrying a foliation $\mathcal{F}$ such that

$$V/\mathcal{F} = \{M_\delta : 0 < \delta \leq \delta_0\}$$

for some $\delta_0 > 0$. Let $H(\mathcal{F}) \to V$ and $T_{1,0}(\mathcal{M}) \to V$ be respectively the bundles whose portions over a leaf M_δ are the Levi distribution $H(M_\delta)$ and the CR structure $T_{1,0}(M_\delta)$ of the leaf. One has

$$T_{1,0}(\mathcal{F}) \cap T_{0,1}(\mathcal{F}) = (0), \quad T_{0,1}(\mathcal{F}) \equiv T_{1,0}(\mathcal{F}),$$

$$Z, W \in C^\infty(T_{1,0}(\mathcal{F})) \Longrightarrow [Z, W] \in C^\infty(T_{1,0}(\mathcal{F})).$$

The rudiments of a general theory of tangentially CR foliations has been already started (cf. e.g. S. Dragomir and S. Nishikawa, [7]). By a result of J.M. Lee and R. Melrose there is a unique complex vector field ξ of type $(1,0)$ on V such that

$$\partial\varphi(\xi) = 1, \quad \partial\overline{\partial}\varphi\,(\xi, \overline{Z}) = 0, \quad Z \in T_{1,0}(\mathcal{F}).$$

The function

$$r = 2\,\partial\overline{\partial}\varphi\,(\xi, \overline{\xi})$$

is the *transverse curvature* of $\mathcal{F}$. The same result by J.M. Lee and R. Melrose (cf. [19]) shows that r is smooth up to the boundary i.e. $r \in C^\infty(\overline{\Omega})$. Let $\xi = \frac{1}{2}(N - iT)$ be the real and imaginary parts of ξ. Then $T(\varphi) = 0$ hence $T \in T(\mathcal{F})$. Also $T \rfloor d\theta = 0$ so that T is transverse to $H(\mathcal{F})$ and actually

$$T(\mathcal{F}) = H(\mathcal{F}) \oplus \mathbb{R}T.$$

Let g_θ be given by

$$g_\theta(X,Y) = (d\theta)(X, JY), \quad g_\theta(X,T) = 0, \quad g_\theta(T,T) = 1.$$

Then g_θ is a tangential Riemannian metric for $\mathcal{F}$ i.e. a Riemannian bundle metric on $T(\mathcal{F}) \to V$ (so that the pullback of g_θ to a leaf of $\mathcal{F}$ is the Webster metric of the leaf). A rather obvious consequence of (16) is that the Bergman metric g of Ω and the tangential metric g_θ are related

$$g(X,Y) = -\frac{n+1}{\varphi}\, g_\theta(X,Y), \quad X, Y \in H(\mathcal{F}), \tag{17}$$

$$g(X,T) = 0, \quad g(X,N) = 0, \quad X \in H(\mathcal{F}), \tag{18}$$

$$g(T,N) = 0, \quad g(T,T) = g(N,N) = \frac{n+1}{\varphi}\left(\frac{1}{\varphi} - r\right). \tag{19}$$

From the point of view of a differential geometer, the meaning (and usefulness) of formulae (17)-(19) is that one may relate, in an effective manner, linear connections on V parallelizing g to linear connections on V parallelizing g_θ. The linear connection parallelizing the Bergman metric one wishes to deal with is the Levi-Civita connection of (Ω, g). Indeed one wishes to study the Bergman-harmonic map equations

$$-\Delta_g \Phi^i + \left(\Gamma^i_{jk} \circ \Phi\right) \frac{\partial \Phi^j}{\partial x^A} \frac{\partial \Phi^k}{\partial x^B} G^{AB} = 0 \tag{20}$$

whose principal part is the Bergman Laplacian

$$\Delta_g u = -\sum_{A=1}^{2n} \left\{ E_A(E_A(u)) - (\nabla^g_{E_A} E_A)(u) \right\}, \quad u \in C^2(\Omega),$$

which is expressed (by using a local g-orthonormal frame $\{E_A : 1 \leq A \leq 2n\}$ of $T(\Omega)$) in terms of covariant derivatives relative to ∇^g, the

Levi-Civita connection of (Ω, g). As to linear connections parallelizing g_θ, these are morally the Tanaka-Webster connections of the leaves of $\mathcal{F}$ (since, as emphasized earlier, the pullback of g_θ to a leaf of $\mathcal{F}$ is the Webster metric of that leaf). Yet what one formally needs is a linear connection on V, rather that on some leaf of $\mathcal{F}$. One is thus led to the question whether one may piece together the Tanaka-Webster connections of the leaves of $\mathcal{F}$. That this is indeed feasible is a result by R. Graham and J.M. Lee (cf. [13]) producing a linear connection ∇ on V whose pointwise restriction to each leaf M_δ is the Tanaka-Webster connection of M_δ. It is the unique linear connection ∇ on V (referred hereafter as the *Graham-Lee connection*) singled out by the following axioms i) $T_{1,0}(\mathcal{F})$ is parallel with respect to ∇, ii) $\nabla L_\theta = 0$, $\nabla T = 0$, $\nabla N = 0$, and iii) the torsion T_∇ is pure i.e.

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2i L_\theta(Z, \overline{W}) T,$$

$$T_\nabla(N, W) r W + i\,\tau(W), \quad Z, W \in T_{1,0}(\mathcal{F}),$$

$$\tau\left(T_{1,0}(\mathcal{F})\right) \subset T_{0,1}(\mathcal{F}), \quad \tau(N) = -J\nabla^H r - 2rT,$$

where

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad \tau(X) = T_\nabla(T, X), \quad X \in T(V).$$

Indeed (17)-(19) may then be used to relate ∇^g to ∇. For instance

$$\nabla^g_X Y = \nabla_X Y + \tag{21}$$

$$+ \left\{ \frac{\varphi}{1 - \varphi r}\, g_\theta(\tau X, Y) + g_\theta(X, \phi Y) \right\} T -$$

$$- \left\{ g_\theta(X, Y) + \frac{\varphi}{1 - \varphi r}\, g_\theta(X, \phi \tau Y) \right\} N$$

for any $X, Y \in H(\mathcal{F})$. Here ϕ is defined by

$$\phi : T(\mathcal{F}) \to T(\mathcal{F}),$$

$$\phi X = JX, \quad \phi T = 0, \quad X \in H(\mathcal{F}).$$

Similar formulae may be got for $\nabla^g_X T$, $\nabla^g_X N$, $\nabla^g_T N$, $\nabla^g_T T$ and $\nabla^g_N N$ hence we express ∇^g as a function of the Graham-Lee connection and the transverse curvature of $\mathcal{F}$ and its first order derivatives

$$\nabla^g = f\left(\nabla,\, r, \nabla^H r,\, T(r),\, N(r)\right). \tag{22}$$

The whole point is that, when we shall analyze (20) as $\varphi \to 0$ (equivalently as $z \to \partial\Omega$), the quantities g_θ, ∇ and τ stay finite at the boundary and give there the Webster metric, the Tanaka-Webster connection, and the pseudohermitian torsion of the boundary. So does r

and its derivatives of any order, by the J.M. Lee and R. Melrose result recalled above. Formula (22) relies on the decomposition

$$T(V) = T(\mathcal{F}) \oplus \mathbb{R}N = H(\mathcal{F}) \oplus \mathbb{R}T \oplus \mathbb{R}N \tag{23}$$

(as (21) already suggests). To exploit said decomposition one chooses a local orthonormal frame $\{W_\alpha : 1 \le \alpha \le n-1\}$ of $T_{1,0}(\mathcal{F})$ i.e. $g_\theta(W_\alpha, W_{\overline{\beta}}) = \delta_{\alpha\beta}$ and sets

$$E_\alpha = \sqrt{-\frac{\varphi}{n+1}}\, W_\alpha\,, \quad E_n = \sqrt{\frac{2\psi\varphi}{n+1}}\, \xi, \quad \psi \equiv \frac{\varphi}{1 - r\varphi}\,.$$

Then $\{E_j : 1 \le j \le n\}$ is a local orthonormal frame of $T^{1,0}(V)$ i.e. $g(E_j, E_k) = \delta_{jk}$, adapted to the decomposition (23) and (22) yields

$$\begin{aligned} \Delta_g = -\frac{\varphi}{n+1}\Delta_b - \frac{2\varphi(n-1)}{n+1}N + \\ + \frac{\psi\varphi}{n+1}\left\{N^2 + T^2 + \nabla^H r + 2rN\right\} \end{aligned} \tag{24}$$

where Δ_b is given by

$$\Delta_b = \sum_{\alpha=1}^{n-1}\left(W_\alpha W_{\overline{\alpha}} + W_{\overline{\alpha}} W_\alpha - \nabla_{W_\alpha} W_{\overline{\alpha}} - \nabla_{W_{\overline{\alpha}}} W_\alpha\right).$$

$$\left(\Gamma^i_{jk} \circ \Phi\right) \frac{\partial \Phi^j}{\partial x^A} \frac{\partial \Phi^k}{\partial x^B} G^{AB} = \tag{25}$$

$$= -\frac{2\varphi}{n+1}\left(\Gamma^i_{jk} \circ \Phi\right)\left\{\sum_{\alpha=1}^{n-1} W_\alpha(\Phi^j) W_{\overline{\alpha}}(\Phi^k) - 2\psi\, \xi(\Phi^j)\, \overline{\xi}(\Phi^k)\right\}.$$

Taking into account (24)-(25) the Bergman-harmonic map system (20) may be written

$$\begin{aligned} \Delta_b \phi^i + 2(n-1)N\phi^i - \psi\left(N^2 + T^2 + \nabla^H r + 2rN\right)\phi^i + \\ +2\left(\Gamma^i_{jk} \circ \phi\right)\left\{\sum_{\alpha=1}^{n-1} W_\alpha(\phi^j) W_{\overline{\alpha}}(\phi^k) - 2\psi \xi(\phi^j)\overline{\xi}(\phi^k)\right\} = 0. \end{aligned} \tag{26}$$

Let $\phi = \phi_f$ be the solution to the Dirichlet problem for the system (20) with the boundary condition $\phi = f$ on $\partial\Omega$ with $f \in C^\infty(\partial\Omega, S)$. Let us assume that $\phi \in C^\infty(\overline{\Omega}, S)$. Then as $\varphi \to 0$ the equation (26) leads to

$$(H_S f)^i + 2(n-1)N f^i = 0, \quad 1 \le i \le \nu.$$

The normal derivatives of the map $f : \partial\Omega \to S$ may thus be determined in terms of purely tangential quantities, a phenomenon looked upon as typically non-elliptic. We close with the statement (cf. [5])

Theorem 2. *Let $\Omega \subset \mathbb{C}^n$ $(n \geq 2)$ be a smoothly bounded strictly pseudoconvex domain and g the Bergman metric on Ω. Let S be a complete ν-dimensional $(\nu \geq 2)$ Riemannian manifold of sectional curvature* $\mathrm{Sect}(S) \leq \kappa^2$ *for some $\kappa > 0$. Assume that S may be covered by one coordinate chart $\chi = (y^1, \cdots, y^\nu) : S \to \mathbb{R}^\nu$. Let $f \in W^{1,2}(\Omega, S) \cap C^0(\overline{\Omega}, S)$ be a map such that $f(\overline{\Omega}) \subset B(p, \mu)$ for some $p \in S$ and some $0 < \mu < \min\{\pi/(2\kappa)\,,\, i(p)\}$ where $i(p)$ is the injectivity radius of p. Let $\phi = \phi_f : \overline{\Omega} \to S$ be the solution to the Dirichlet problem*

$$\tau_g(\phi) = 0 \quad \text{in } \Omega, \quad \phi = f \quad \text{on } \partial\Omega. \tag{27}$$

If $f \in C^\infty(\partial\Omega\,,\, S)$ and $\phi \in C^\infty\left(\overline{\Omega}, S\right)$ then

$$N(f^i) = -\frac{1}{2(n-1)}\,(H_S f)^i\,, \quad 1 \leq i \leq \nu, \tag{28}$$

for any local coordinate system (ω, y^i) on S such that $\phi(\overline{\Omega}) \cap \omega \neq \emptyset$ $(f^i = y^i \circ f)$. Also $N = -JT$ and T is the characteristic direction of $\partial\Omega$. In particular if $N(f^i) = 0$ then $f : \partial\Omega \to S$ is a subelliptic harmonic map.

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