The Existence of CR Structures
Workshop on Geometric Analysis of PDEs and Several Complex Variables
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The most important example of a CR manifold:

\[ M^{2n+1} \subset \mathbb{C}^{n+1} \]

\[
B = T^{0,1} \cap (\mathbb{C} \otimes T(M))
\]

\[
= \{ L = \sum_{1}^{n+1} \alpha_j \frac{\partial}{\partial z_j} \mid L = X + iY, \ X \in TM, \ Y \in TM \}
\]

Properties

- \( \text{rank } B = n \)
- \( B \cap \overline{B} = \{0\} \)
- \( [B, B] \subset B \)
Abstract CR structure

$B \subset C \otimes T(M)$ is a CR structure of codimension $k$ on $M^{2n+k}$ if
- rank $B = n$
- $B \cap \overline{B} = \{0\}$
- $[B, B] \subset B$

Example If $M^{2n+k} \subset \mathbb{C}^{n+k}$ is “generic” at $p \in M$, then near $p$

$$B = T^{0,1} \cap (C \otimes T(M))$$

is a CR structure of codimension $k$.

Digression Not every abstract CR structure is realizable.
When does $M^{2n+k}$ admit a CR structure of codimension $k$? Need that $M$ admits an almost complex structure:

- $\text{rank } B = n$
- $B \cap \overline{B} = \{0\}$
- $[B, B] \subset B$

The question becomes

When can $B$ be deformed to a CR structure? Seek $B_t \subset C \otimes T(M)$, for $0 \leq t \leq 1$ with

$$\text{rank } B_t = n$$

$B_0 = B$

$B_t \cap \overline{B}_t = \{0\}$
Conjecture

Let $M$ be an open and orientable manifold. If the cohomology groups $H^q(M, \mathbb{Z})$ vanish for $q$ large (depending on $k$), then every almost CR structure of codimension $k$ may be deformed to a CR structure.

Known results

\[
\begin{align*}
    k &= 0 & q \geq \frac{\dim M}{2} \\
    k &= \dim M - 2 & q \geq \dim M + 1 \text{ (That is, there is no cohomological restriction.)}
\end{align*}
\]
Related questions

- Can every CR structure on $M$ be approximated by a $C^\omega$ structure?
- Can every CR structure on an open subset of $M$ be extended to a CR structure on all of $M$?
- Can a non-degenerate CR structure on an open subset of $M$ be extended to a CR structure of the same signature on all of $M$?
Heuristic motivation

Observe

If $M^{2n+k}$ has a map into $C^{n+k}$ that is generic at all points of $M$, then

$$B = f_*(C \otimes T(M)) \cap T^{0,1}$$

determines a CR structure of codimension $k$.

Given some $B^n \subset C \otimes T(M)$ would like to find some complex manifold $X^{n+k}$ and a generic map

$$f : M \to X.$$ 

Actually, $X$ will be of real dimension $4n + 3k$ and foliated by complex manifolds of complex dimension $n + k$.

$X$ is used to provide half of the proof of the conjecture:
Theorem

Let $B$ be a continuous almost CR structure of codimension $k$ on $M^{2n+k}$. If $(C \otimes T(M))/B$ is isomorphic to the normal bundle of a Haefliger CR structure then $B$ is homotopic through almost CR structures to a $C^\omega$ CR structure.
Approach modeled on classical foliation theory

Analysts A foliation means the Frobenius Theorem: A subbundle $K \subset TM$ defines a foliation if and only if $[K, K] \subset K$.

Topologists A foliation of codimension $q$ is a collection of open sets

$$M = \bigcup \mathcal{O}_i,$$

submersions onto open subsets of $\mathbb{R}^q$

$$f_i : \mathcal{O}_i \to U_i,$$

and compatibility conditions.
Haefliger: To show that any $K \subset TM$ may be deformed to the tangent bundle of a foliation

**Step 1** If $TM/K$ is isomorphic to the normal bundle of a Haefliger structure then the Gromov-Phillips Theorem may be used to find the foliation.

**Step 2** Find topological conditions on $M$, depending on $\text{rank}K$, such that for all $K$, $TM/K$ is isomorphic to such a normal bundle.
Gromov-Phillips Theorem

Simplest case

\[ f : \quad M \to X \]
\[ g : \quad TM \to TX \]

with

\[ g : \quad TM_p \to TX_{f(p)} \]

surjective at each \( p \in M \).

**Theorem**

There exists a submersion \( F : M \to X \).
Haefliger Structures

An example and then the definition.

If $M^{2n+k}$ has a $C^\omega$ CR structure of co-dimension $k$ then there exists an open covering

$$M = \bigcup O_j$$

and $C^\omega$ CR embeddings

$$f_j : O_j \rightarrow \mathbb{C}^{n+k}.$$ 

Further, for each pair $i, j$ with $O_i \cap O_j \neq \emptyset$ there exist open sets $U_{ij}$ containing $f_i(O_i \cap O_j)$ and biholomorphism $\gamma_{ij} : U_{ji} \rightarrow U_{ij}$ with

$$f_i = \gamma_{ij} \circ f_j \text{ on } O_i \cap O_j.$$ 

and

$$\gamma_{ik} = \gamma_{ij} \circ \gamma_{jk}.$$
Definition

A Haefliger CR structure of codimension $k$ on $M^{2n+k}$ consists of

- An open covering $M = \bigcup \mathcal{O}_j$,
- continuous maps $f_j : \mathcal{O}_j \to \mathbb{C}^{n+k}$,
- local biholomorphisms $\gamma_{ij}$ of $\mathbb{C}^{n+k}$ defined for each pair $(i,j)$ such that $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$ satisfying
  \[ \gamma_{ik} = \gamma_{ij} \circ \gamma_{jk} \]
  at all points where both sides are defined and
  \[ f_i = \gamma_{ij} \circ f_j \text{ on } \mathcal{O}_i \cap \mathcal{O}_j. \]
The normal bundle \( \nu \) of a Haefliger CR structure is the \( \mathbb{C}^{n+k} \) bundle over \( M \) with transitions functions \( d\gamma_{ij} \).

**Lemma**

*The normal bundle \( \nu \) of a Haefliger CR structure admits in a neighborhood of the zero section a foliation of dimension \( 2n + k \) transverse to the bundle fibers.*

Micro-foliation
Step 1 for CR Structures

We assume $C \otimes T(M)/B$ is isomorphic to the normal bundle of a Haefliger structure and find a CR structure. Write $X$ in place of $\nu$.

- $\dim X = 4n + 3k$.
- $X$ has transverse foliations $\mathcal{F}^{2n+2k}$ and $\mathcal{F}^{2n+k}$.
- The leaves of $\mathcal{F}^{2n+2k}$ are complex manifolds $V^{n+k}$.

1- Use $\mathcal{F}^{2n+k}$ to define

$$p : C \otimes T(X) \to T^{1,0}V.$$ 

2- Use $p$ and $C \otimes T(M)/B \cong \nu$ to construct a surjective map

$$C \otimes T(M) \to T^{1,0}V|_M$$

with kernel $B$. 
3- Use the h-principle to find a map

\[ F : M \rightarrow X \]

with

\[ p \circ F_* : C \otimes T(M) \rightarrow T^{1,0}V \]
surjective.

4- Observe that \( B = \ker p \circ F_* \) satisfies

- \( \text{rank } B = n \)
- \( B \cap \overline{B} = \{0\} \)
- \([B, B] \subset B\).
Step 2 for CR Structures

\[ \mathcal{B} \oplus GL(n) \]
\[ \downarrow \Gamma(\nu_B) \oplus \text{id} \]
\[ M \xrightarrow{\Gamma(\nu) \oplus \Gamma(B)} BGL(n + k) \oplus BGL(n) \]
The obstructions to lifting

\[ Y \]
\[ \downarrow \]
\[ M \rightarrow X \]

lie in

\[ H^{i+1}(M, \pi_i(F)). \]

So if

\[ \pi_j(F) = 0 \]

for \( 0 \leq j \leq \dim M \), and if

\[ H^j(M, Z) = 0 \]

for \( N + 2 \leq j \leq \dim M \), then all maps \( M \rightarrow X \) lift.