CR Structures on Open Manifolds

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Theorem (with P. Landweber)

If

$$H_p(M^{2n+k}; \mathbf{Z}) = 0 \text{ for } p \geq n + k + 1$$

then every smooth almost CR structure of codimension k on M is homotopic to a C^ω CR structure of codimension k.

Let M be a manifold of dimension $2n + k$ with $k > 0$.

An almost CR structure of codimension k on M is a complex subbundle $B \subset \mathbf{C} \otimes T(M)$ of complex rank n that satisfies $B \cap \overline{B} = \{0\}$.

A CR structure of codimension k on M is an almost CR structure B of codimension k that in addition is involutive.

$$[B, B] \subset B$$

Example

A generic immersion is an immersion f

$$f : M^{2n+k} \to \mathbf{C}^{\mathbf{n}+\mathbf{k}}$$

such that

$$(\mathbf{C} \otimes T(M)) \cap f^* T^{0,1}(\mathbf{C}^{\mathbf{n}+\mathbf{k}})$$

has complex rank n at all points of M.

Then

$$B = (\mathbf{C} \otimes T(M)) \cap f^* T^{0,1}(\mathbf{C}^{\mathbf{n}+\mathbf{k}})$$

defines a CR structure on M.

When this agrees with a given CR structure, we say that f is a CR immersion.

Lemma

If M has a C^ω CR structure B of codimension k then there exists an open covering

$$M = \bigcup_j \mathcal{O}_j$$

and C^ω generic CR embeddings

$$f_j : \mathcal{O}_j \to \mathbf{C}^{n+k}.$$

Further, for each pair (i, j) with $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$ there exists an open set U_{ij} containing $f_i(\mathcal{O}_i \cap \mathcal{O}_j)$ and a biholomorphism $\gamma_{ij} : U_{ji} \to U_{ij}$ with

$$f_i = \gamma_{ij} \circ f_j \text{ on } \mathcal{O}_i \cap \mathcal{O}_j.$$

Follow the approach of topologists to foliations.

1. Define an HCR by weakening the definition of a CR structure.
2. If M admits an HCR and if $\mathbf{C} \otimes T(M)/B$ is isomorphic to a bundle associated to the HCR, then M admits a CR structure.
3. Topological conditions on M imply that these hypotheses are satisfied.

Foliation of co-dimension q

1. An open covering $M^{m+q} = \bigcup_j \mathcal{O}_j, \;\; j \in A$,
2. smooth maps $f_j : \mathcal{O}_j \to \mathbf{R}^\mathbf{q}$,
3. local diffeomorphisms γ_{ij} of $\mathbf{R}^\mathbf{q}$ defined for each pair (i, j) such that $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$ satisfying:

$$f_i = \gamma_{ij} \circ f_j \text{ on } \mathcal{O}_i \cap \mathcal{O}_j.$$

Haefliger foliation

1. An open covering $M^{m+q} = \bigcup_j \mathcal{O}_j, \ \ j \in A$,
2. continuous maps $f_j : \mathcal{O}_j \to \mathbf{R^q}$,
3. local diffeomorphisms γ_{ij} of $\mathbf{R^q}$ defined for each pair (i, j) such that $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$ satisfying:

$$f_i = \gamma_{ij} \circ f_j \text{ on } \mathcal{O}_i \cap \mathcal{O}_j.$$

4. $$\gamma_{ij} = \gamma_{ik}\gamma_{kj}$$

Typical theorem from foliation theory.

Let M be compact. If M has a trivial sub-bundle of rank q then it has a q codimension foliation.

Haefliger CR structure (HCR structure)

1. An open covering $M = \bigcup_j \mathcal{O}_j, \;\; j \in A$,
2. continuous maps $f_j : \mathcal{O}_j \to \mathbf{C}^{n+k}$,
3. local biholomorphisms γ_{ij} of $\mathbf{C}^{n+k}$ defined for each pair (i, j) such that $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$ satisfying:
4. $f_i = \gamma_{ij} \circ f_j$ on $\mathcal{O}_i \cap \mathcal{O}_j$.
5. $\gamma_{ik} = \gamma_{ij} \circ \gamma_{jk}$ at all points where both sides are defined.

If M^{2n+k} admits an HCR then there exists some X^{4n+3k}

$$\begin{array}{c} X \\ \Big\downarrow \pi \\ M \end{array}$$

such that each fiber $\pi^{-1}(p)$ is a complex manifold and a section $\iota : M \to X$.

Want to perturb ι to some immersion $F : M \to X$ such that the composite map

$$\mathbf{C} \otimes T(M) \to \mathbf{C} \otimes T(X) \to T_f^{1,0}$$

is surjective.

Then $\ker \mu \mathcal{F}_*$ is a co-dimension q CR structure on M.

General philosophy:
To find a map

$$F : M \to X$$

with certain properties, start with a map

$$G : TM \to TX$$

with the corresponding properties.

h-principle (Thom?, Smale, Phillips, Gromov)

Associated to HCR there is a normal bundle, ν.

The transition functions for this normal bundle are

$$g_{ij} = d\gamma_{ij}.$$

Theorem

If $\mathbf{C} \otimes T(M)/B$ *is isomorphic to the normal bundle* ν *of a HCR structure, then* B *can be deformed to a CR structure.*

Note that the hypothesis implies that M admits a HCR structure.

Why does the vanishing of the higher homology groups imply

$$\mathbf{C} \otimes T(M)/B \cong \nu?$$

Recall the classifying space $BGL(m)$ for vector bundles of rank m over paracompact manifolds.

$$\{\text{ isomorphism classes of vector bundles of rank } m\} \xrightarrow{\mathrm{cl}\,(\nu)} [M, BGL(m)]$$

There is also a classifying space for HCR structures, $\mathcal{B}_{n,k}$ which has its own normal $\nu_{n,k}$.

Let $\nu = TM/B$ and write $TM = \nu \oplus B$.

$$\begin{array}{ccc} & & \mathcal{B}_{n,k} \times BGL(n) \\ & & \Big\downarrow \mathrm{cl}\,(\nu_{n,k}) \times \mathrm{id} \\ M & \xrightarrow{\mathrm{cl}\,(\nu) \times \mathrm{cl}\,(B)} & BGL(n+k) \times BGL(n). \end{array}$$

We need to lift the bottom arrow and thereby show that ν is the normal bundle of some HCR structure.

The obstructions to lifting lie in $H^{j+1}(M, \pi_j(\mathcal{F}))$.

Known result

$$\pi_j(\mathcal{F}) = 0 \quad \text{for} \quad 0 \leq j \leq n+k.$$

So we need

$$H^{n+k+1+i}(M, \pi_{n+k+i}(\mathcal{F})) = 0 \quad 0 \leq i.$$

This holds, provided

$$H_p(M^{2n+k}; \mathbf{Z}) = 0 \;\text{ for }\; p \geq n+k+1$$

as a consequence of the Universal Coefficient Theorem.

Open Questions

Let B be an almost CR structure. We seek conditions under which we can prove more than the existence of deformations into CR structures.

Or find obstructions to these improvements.

1- If B is already CR on some open set, can the deformation be taken to leave B unchanged on at least a slightly smaller open set?

2- If B is strongly pseudoconvex, can the deformation be taken to produce a strongly pseudoconvex CR structure?

And best of all

3- Can we do both of these simultaneously?

If so then every five dimensional strictly pseudoconvex CR structure is

locally realizable.

Another problem to show existence or find obsructions:

Can every smooth CR structure on a manifold be deformed to become a real analytic CR structure? Can the deformation be taken to be small?