# OPTIMALITY FOR SOME TARGET DIMENSIONS COLLOQUIUM LECTURE UNIVERSIDAD DE GUANAJUATO DECEMBER 12, 2012

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Sometimes theorems can be proved by just counting dimensions.

The prototype is: If X and Y submanifolds of Z then there exists a perturbation of X such that X∩ Y=∅ provided dim X+dim Y < dim Z.

Want to do three examples of such theorems and focus on whether the results obtained by such simple arguements are in fact optimal. Somewtimes they are, sometimes they are not.

**Theorem 1** (Whitney 1936). *If $V$ is a manifold and if $N \geq 2dim\ V$ then there is a smooth immersion of $V$ into $R^N$. Also if $f : V \to R^N$ is any smooth map, then there is a smooth immersion of $V$ into $R^N$ which approximates $f$.*

*Proof.* In terms of local coordinates,

$$f : V \to R^N$$

has as its jet

$$j^1(f) = (p, f(p), \frac{\partial f^j}{\partial v_i})$$

where $1 \leq j \leq N$ $1 \leq i \leq \dim V$.

The condition that $f$ is an immersion at $p$ is

$$\text{rank } (\frac{\partial f^a}{\partial v_b}) = \dim V.$$

So the singular set $\Sigma \subset J^1$ is,

$$\{(p, c, A) \ : \ p \in V, \ c \in W, \ A \in M(N, \dim V), \ \text{rank } A < \dim V\}.$$

That is,

$$j^1_p(f) \in \Sigma_p \text{ if and only if } f \text{ is not an immersion at } v.$$

We have this situation. Start with any $f : V \to R^N$. This gives a subset $j^1(f)V \subset J^1$.

$$dim j^1(f)V = \dim V$$

so $\dim V < codim \Sigma$ implies can perturb $j^1(f)V$ to avoid $\Sigma$.

**Theorem 2** (Thom Transversality Theorem)**.** *Let* $\Sigma$ *be a stratified subset of* $J^1(V,W)$*. The set of maps* $F : V \to W$ *with* $j^1(F)$ *intersecting* $\Sigma$ *transversally is generic.*

So we need to compute the codim of $\Sigma$.

**Lemma 1.** *Let* $M(m,n)$ *be the set of* $m \times n$ *matrices with real coefficients. The set*

$$\Sigma = \{g \in M(m,n) \mid \operatorname{rank} g < \min(m,n)\} \tag{1}$$

*is a stratified subset of* $M(m,n)$ *of codimension* $\max(m,n) - \min(m,n) + 1$*.*

We write

$$\Sigma = \cup \Sigma^r$$

To prove this, we start with a single stratum.

**Lemma 2.** *Let* $r \leq \min(m,n)$*. The set*

$$\Sigma^r = \{g \in M(m,n) \mid \operatorname{rank} g = r\}$$

*can be defined locally as the zero set of* $(n-r)(m-r)$ *independent functions.*

To prove Whitney theorem it remains to take

$$\dim V < codim \Sigma = N - \dim V + 1$$
$$2 \dim V < N + 1$$
$$2 \dim V \leq N$$

□

**Theorem 3** (Whitney 1944)**.** *Each smooth map* $f : V^n \to \mathbf{R}^{2n-1}$ *may be approximated by a smooth immersion.*

This result is optimal for some dimensions.

**Theorem 4.** *There is no immersion of* $\mathbf{RP}^n$ *into* $\mathbf{R}^{2n-2}$ *when* $n$ *is of the form* $n = 2^k$*.*

*Proof.* We follow section 4 of Milnor Stashof. The proof relies on the basic properties of the Steifel-Whitney characteristic classes. Given a real bundle $B$ over a manifold $M^n$, the Steifel-Whitney class $w(B)$ is a sum of mod 2 cohomology classes

$$w(B) = 1 + w_1(B) + w_2(B) + \ldots + w_n(B)$$

with

$$w_i(B) \in H^i(M, Z_2).$$

Given two such bundles $B_1$ and $B_2$ over $M$ we have

$$w(B_1 \oplus B_2) = w(B_1)w(B_2)$$

with the multiplication given by the cup product in the cohomology ring. The notation $w(M)$ is used for $w(TM)$.

The Steifel-Whitney classes for projective space is given by

$$w(\mathbf{RP}^n) = (1 + a)^{n+1}$$

where $a = w_1(\gamma)$ and $\gamma$ is the canonical line bundle on $\mathbf{RP}^n$.

If we have an immersion

$$\mathbf{RP}^n \to \mathbf{R}^N$$

with normal bundle $\nu$, then $T\mathbf{RP}^n \oplus \nu$ is a trivial bundle over $\mathbf{RP}^n$ and hence

$$(1 + a)^{n+1} w(\nu) = 1.$$

For $n = 2^k$, most of the binomial coefficients in $(1 + a)^{n+1}$ are zero, mod 2. This, together with $a^{n+1} = 0$, since it belongs to $H^{n+1}(M^n, Z_2)$, gives

$$(1 + a)^{n+1} = 1 + a + a^n.$$

Let

$$w(\nu) = 1 + \alpha_1 + \ldots + \alpha_{N-n}.$$

From

$$(1 + a + a^n)(1 + \alpha_1 + \ldots + \alpha_{N-n}) = 1$$

we see that for some $j$, $1 \leq j \leq N - n$, we need to have

$$a\alpha_j = a^n.$$

In particular, we need

$$1 + N - n \geq n.$$

That is,

$$N \geq 2n - 1.$$

So $\mathbf{RP}^{2^k}$ cannot be immersed into $\mathbf{R}^{2^k-2}$. □

## 1. Totally Real Immersions $M \to \mathbf{C}^N$

Recall that the complex vector space $\mathbf{C}^N$ can be identified with $(\mathbf{R}^{2N}, J)$ where $J$ satisfies $J^2 = -I$. In terms of the usual coordinates for the real vector space underlying $\mathbf{C}^N$

$$J\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}, \qquad J\frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}.$$

We shorten this to $J\partial x_k = \partial y_k$, etc.

**Definition 1.** *A submanifold $M \subset \mathbf{C}^N$ is totally real if*

$$TM \cap JTM = \{0\}. \tag{2}$$

Note that the dimension of $M$ must be less than or equal to $N$. The simplest example is, of course, the standard embedding of $\mathbf{R}^N$ into $\mathbf{C}^N$.

A map

$$F : M \to \mathbf{C}^N$$

is a totally real immersion if

(1) $F$ is an immersion and
(2) $F_*TM \cap JF_*TM = \{0\}$.

Since $M^n$ embeds into $\mathbf{R}^{2n} \subset \mathbf{C}^{2n}$ and any submanifold of a totally real submanifold is itself totally real, we have for free that $M^n$ always has a totally real embedding into $\mathbf{C}^{\mathbf{N}}$ provided $N \geq 2n$.

**Theorem 5.** *There exists a totally real immersion of $M^n$ into $\mathbf{C}^N$ provided $N > \frac{3}{2}n - 1$.*

**Remark 1.** *We actually get a totally real embedding.*

Let

$$\Sigma \subset J^1(M, \mathbf{C}^N) = J^1(M, \mathbf{R}^{2N})$$

be defined by the fibers

$$\Sigma_p = \{j^1_p(f) |\ f : M^n \to \mathbf{C}^N \text{ with } f_*TM_p \cap Jf_*TM_p \neq \{0\}\}.$$

Any immersion that avoids $\Sigma$ is a totally real immersion. For this we show that $\Sigma$ is a stratified manifold of codimension greater than $n$. To start, we give an alternate description of $\Sigma_p$. Choose a local coordinate system containing $p$. Let $f : M^n \to \mathbf{R}^{2N}$ and let $A(f) \in M(2N, n))$ be given by

$$A_{ij} = \frac{\partial f^j}{\partial x_i}(p).$$

Denote by $(A, JA)$ the element of $M(2N, 2n)$ obtained by juxtaposition of the matrices $A$ and $JA$. Then $f$ is a totally real immersion at $p$ if and only if

$$rank(A, JA) = 2n.$$

$$\Sigma = \{(p, c, A) \ : \ rank(A, JA) \leq 2n\}.$$

The codim of $\Sigma$ is $2N - 2(n-1)$ . The codimension of $\Sigma$ is greater than the dimension of $M$ when $N > 3n/2 - 1$.

## 2. Complex-valued Functions

**Definition 2.** *A map $f : M^n \to \mathbf{C}^N$ is called an **independent map** if*

$$df_1(p) \wedge \cdots \wedge df_N(p) \neq 0$$

*for $f = (f_1, \ldots, f_N)$ and for all $p \in M$.*

**Theorem 6.** *Any map $f : M^n \to \mathbf{C}^N$ may be approximated by an independent map, provided $N \leq [\frac{n+1}{2}]$.*

Now we prove Theorem **??**. It is convenient to use a slightly different representation of the space $J^1(M, \mathbf{C}^r)$ of one-jets of maps $F : M^n \to C^r$ by writing a local section $j^1(F)$ as $\{(p, F(p), dF_1(p), \ldots, dF_r(p))\}$ and a point of $J^1(M, \mathbf{C}^r)$ as $(p, c, \theta_1, \ldots, \theta_r)$.

Let $\Sigma$ be the subset of $J^1(M, \mathbf{C}^r)$whose fiber at each point $p$ is given by $\{(p, c, \theta) : \theta_1 \wedge \ldots \wedge \theta_r|_p = 0\}$.

**Lemma 3.** *$\Sigma$ is a stratified subset of $J^1(M, \mathbf{C}^r)$ of codimension $2(n + 1 - r)$.*

The codimension of $\Sigma$ in $J^1(M, \mathbf{C}^r)$ is greater than the dimension of $M^n$ provided we require $2(n + 1 - r) > n$. It follows from the Thom Transversality Theorem that any map of $M$ into $\mathbf{C}^r$ may be perturbed to achieve that its one-jet does not intersect $\Sigma$.

## 3. Optimality

To explain our examples, we find necessary bundle-theoretic conditions for totally real immersions and for independent maps.

**Lemma 4.** (a) *If $M$ has a totally real immersion into $\mathbf{C}^N$ then there exists a bundle $Q$ of rank $r = N - n$ such that*

$$(\mathbf{C} \otimes TM) \oplus Q \cong N\varepsilon.$$

(b) *If $M$ has an independent map into $\mathbf{C}^N$ then there exists a bundle $B$ of rank $r = n - N$ such that*

$$\mathbf{C} \otimes TM \cong N\varepsilon \oplus B.$$

Here $N\varepsilon$ is the product complex vector bundle over $M$ of rank $N$.

*Proof of Lemma 4.* (a) The condition $TM \cap JTM = \emptyset$ is equivalent to the fiber injectivity of

$$\phi_f : \mathbf{C} \otimes TM \to T^{1,0}(\mathbf{C}^N)$$

where $\phi_f(v)$ is defined, for $v \in \mathbf{C} \otimes TM$, by

$$\phi_f(v) = f_*(v) - iJf_*(v).$$

Thus if $M$ has a totally real immersion into $\mathbf{C}^N$ then

$$(\mathbf{C} \otimes TM) \oplus Q \cong N\varepsilon$$

where $Q$ is the bundle in $T^{1,0}$ normal to $\phi_f(\mathbf{C} \otimes TM)$.

(b) The map

$$\psi_f : \mathbf{C} \otimes TM \to T^{1,0}$$

given by

$$\psi_f(v) = \sum df_j(v)\partial_{z_j}$$

is surjective on the fibers. So

$$\mathbf{C} \otimes TM \cong N\varepsilon \oplus B.$$

with $B = \ker \psi_f$. □

3.1. **Totally real immersions.** We need to find a manifold of dimension $n$ that does not have a totally real immersion into $\mathbf{C}^N$ for $N = [\frac{3n}{2}] - 1$. We provide four families of examples according to the residue of the dimension of $M$ modulo 4. Let

$$M^{4k} = \mathbf{C}\mathrm{P}^2 \times \cdots \times \mathbf{C}\mathrm{P}^2 = (\mathbf{C}\mathrm{P}^2)^{\times k}$$

be the product of $k$ copies of the complex projective plane.

**Theorem 7** (with Landweber 2012)**.**

- *$M^{4k}$ does not admit a totally real immersion into $\mathbf{C}^N$ for $N = 6k-1$.*
- *$M^{4k+1} = M^{4k} \times S^1$ does not admit a totally real immersion into $\mathbf{C}^N$ for $N = 6k$.*
- *$M^{4k+2} = M^{4k} \times \mathbf{R}\mathrm{P}^2$ does not admit a totally real immersion into $\mathbf{C}^N$ for $N = 6k+2$.*
- *$M^{4k+3} = M^{4k} \times \mathbf{R}\mathrm{P}^2 \times S^1$ does not admit a totally real immersion into $\mathbf{C}^N$ for $N = 6k+3$.*

Denote the total Chern class of a complex vector bundle $B$ over $M$ by

$$c(B) = 1 + c_1(B) + \cdots + c_k(B)$$

where $c_j(B) \in H^{2j}(M; \mathbf{Z})$ and $k = \min(\operatorname{rank} B, [\frac{\dim M}{2}])$.

We will use

$$c(E \oplus F) = c(E)c(F).$$

We have the following well-known result (see e.g. [**?**, Section 14]).

**Lemma 5.** *Let $a$ denote the first Chern class of the hyperplane line bundle $\mathcal{O}(1)$ on $\mathbf{C}\mathrm{P}^2$. Then*

$$c(\mathbf{C} \otimes T\mathbf{C}\mathrm{P}^2) = 1 - 3a^2.$$

We need to show that in the first two cases of Theorem 7 there is no bundle $Q$ of rank $2k-1$ and in the last two cases no bundle $Q$ of rank $2k$ such that $(\mathbf{C} \otimes TM) \oplus Q$ is trivial. We shall show this for $M^{4k+1}$ and $M^{4k+3}$. The other two cases, which are very similar to these, are done in [**?**]. So first we assume that there is some $Q$ with

$$(\mathbf{C} \otimes TM^{4k+1}) \oplus Q \cong N\varepsilon$$

for some $N$ and show that the rank of $Q$ is at least $2k$.

Let $a_1, \ldots, a_k$ be the pull-backs of $a$ to $M$ under the corresponding projections to $\mathbf{C}\mathrm{P}^2$, so that $a_i^3 = 0$ for all $i$. We have

$$c(\mathbf{C} \otimes TM^{4k+1}) \cdot c(Q) = 1.$$

Thus $c(Q) = (1 + 3a_1^2) \cdots (1 + 3a_k^2)$. Since $a_1^2 \cdots a_k^2 \neq 0$, this implies that the rank of $Q$ is at least $2k$.

Next we assume that there exists some $Q$ with

$$(\mathbf{C} \otimes TM^{4k+3}) \oplus Q = N\varepsilon$$

for some $N$ and show that the rank of $Q$ is at least $2k+1$. Let $a_1, \ldots a_k$ be as before and let $b_1$ be the pull-back of the generator in $H^2(\mathbf{R}\mathrm{P}^2; \mathbf{Z})$ given by the Chern class of the complexification of the tautological line bundle on $\mathbf{R}\mathrm{P}^2$. We have

$$c(\mathbf{C} \otimes TM^{4k+3}) \cdot c(Q) = 1$$

which now gives

$$c(Q) = (1 + 3a_1^2) \cdots (1 + 3a_k^2)(1 - b_1).$$

This implies that the rank of $Q$ is at least $2k + 1$.

3.2. **Independent maps.** The same manifolds $M^{4k+r}$ $(0 \leq r \leq 3)$ show that Theorem 6 is also optimal.

**Theorem 8** (with Landweber)**.**

- *$M^{4k}$ does not admit an independent map into $\mathbf{C}^N$ for $N = 2k + 1$.*
- *$M^{4k+1} = M^{4k} \times S^1$ does not admit an independent map into $\mathbf{C}^N$ for $N = 2k + 2$.*
- *$M^{4k+2} = M^{4k} \times \mathbf{R}\mathrm{P}^2$ does not admit an independent map into $\mathbf{C}^N$ for $N = 2k + 2$.*
- *$M^{4k+3} = M^{4k} \times \mathbf{R}\mathrm{P}^2 \times S^1$ does not admit an independent map into $\mathbf{C}^N$ for $N = 2k + 3$.*

The proofs are similar to those of Theorem 7 and can be found in [**?**]. For instance, to show that $M^{4k+1}$ does not admit an independent map into $\mathbf{C}^N$ for $N = 2k + 2$ we start with

$$\mathbf{C} \otimes TM^{4k+1} \cong N\varepsilon \oplus B$$

for some $N$ which gives us

$$c(B) = c(\mathbf{C} \otimes TM^{4k+1}) = (1 - 3a_1)^2 \cdots (1 - 3a_k^2).$$

So the rank of $B$ is at least $2k$ and since $N + \operatorname{rank} B = 4k + 1$, this leads to $N \leq 2k + 1$.

**Remark 2.** We provide a summary of the results in this section for a four-dimensional manifold $M$, in terms of the following list of conditions that the manifold may satisfy:

(1) $M$ admits a totally real immersion into $\mathbf{C}^5$.
(2) $M$ admits a totally real immersion into $\mathbf{C}^4$.
(3) $M$ admits an independent map into $\mathbf{C}^3$.
(4) $M$ admits an independent map into $\mathbf{C}^4$.
(5) $\mathbf{C} \otimes TM$ is trivial.
(6) The first dual Pontryagin class of $M$ vanishes.
(7) The first Pontryagin class of $M$ vanishes.

Then:

(a) Conditions (2), (4), and (5) are equivalent for all 4-manifolds, and plainly imply the remaining conditions.
(b) Conditions (1) and (6) are equivalent for all 4-manifolds. The same holds for Conditions (3) and (7).
(c) Conditions (1), (3), (6), and (7) are all satisfied if $M$ is open.
(d) All seven conditions are equivalent if $M$ is orientable.
(e) By Theorem 4.2, conditions (1) and (3) are not equivalent for compact non-orientable manifolds; indeed, neither implies the other.
(f) The conditions (1), (3), (6), and (7) are satisfied by the non-orientable manifolds $\mathbf{R}\mathrm{P}^2 \times \mathbf{R}^2$ and $\mathbf{R}\mathrm{P}^2 \times S^2$, but these manifolds do not satisfy the conditions (2), (4), and (5), since in both case the first Chern class of the complexified tangent bundle is nonzero.

It seems unlikely that such complete results can be obtained for manifolds of larger dimension.