CR Structures on Open Manifolds

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Theorem (with P. Landweber)

If

$$H_p(M^{2n+k}; \mathbf{Z}) = 0 \text{ for } p \geq n + k + 1$$

then every smooth almost CR structure of codimension k on M is homotopic to a C^ω CR structure of codimension k. In particular, every C^∞ CR structure may be deformed to a C^ω CR structure.

Question 1 Is every five dimensional strictly pseudoconvex CR structure locally realizable?

Every strictly pseudoconvex CR structure on a compact manifold of dimension five or higher is locally realizable. (Boutet de Monvel, 1975)

In particular, does every strictly pseudoconvex CR structure in the neighborhood of a point of S^5 extend to all of S^5 as a strictly pseudoconvex structure?

Definition

A system of vector fields $L - 1, \ldots, L_n$ is aberrant on a domain Ω if at every point of Ω the only germs f satisfying

$$L_j f = 0 \text{ for } j = 1, \ldots, n$$

are the constants.

Jacobowitz-Treves (1983) claimed there exist aberrant CR structures of Levi signature $(n-1, 1)$. Further, we claimed that the aberrant structures are dense.

Theorem

For each CR structure L of signature $(n-1,1)$ and each point of Ω there exists a CR structure L' agreeing with it to infinite order at that point and such that each germ satisfying

$$L'_j f = 0 \text{ for } j = 1, \ldots, n$$

must also satisfy $df = 0$ at the point.

Question 2: Can every CR structure of a given signature in a neighborhood of the origin in R^{2n+1} be extended to a CR structure on all of R^{2n+1} of the same signature?

Question 3: Can every smooth CR structure on a manifold be perturbed to become real analytic?

Yes, if the manifold satisfies the homology conditions of the Theorem.

If

$$H_p(M^{2n+k}; \mathbf{Z}) = 0 \ \text{ for } \ p \geq n + k + 1$$

then there exists

1. An open covering $M = \bigcup_j \mathcal{O}_j, \ \ j \in A$,
2. continuous maps $f_j : \mathcal{O}_j \to \mathbf{C}^{n+k}$,
3. local biholomorphisms γ_{ij} of $\mathbf{C}^{n+k}$ defined for each pair (i, j) such that $\mathcal{O}_i \cap \mathcal{O}_j \neq \emptyset$ satisfying:
    - $\gamma_{ik} = \gamma_{ij} \circ \gamma_{jk}$ at all points where both sides are defined, and
    - $f_i = \gamma_{ij} \circ f_j$ on $\mathcal{O}_i \cap \mathcal{O}_j$.

$$
\begin{array}{ccc}
 & & \mathcal{B}_{n,k} \times BGL(n) \\
 & \nearrow & \Big\downarrow \mathrm{cl}\,(\nu_{n,k}) \times B(\mathrm{id}) \\
M & \xrightarrow{\mathrm{cl}\,(\nu)\times\mathrm{cl}\,(B)} & BGL(n+k) \times BGL(n).
\end{array}
$$

Lemma

Under the homological conditions on M^{2n+k} there exists a manifold X of real dimension $4n+3k$ and an embedding $\iota : M \to X$ such that

1. *X is a fiber bundle over M with complex structure on the fibers.*
2. *X admits a foliation $\mathcal{F}^{2n+k}$ transverse to the fibers.*
3. *There is a surjective bundle map*

$$\mathbf{C} \otimes TM \to \iota^* T_f^{1,0}$$

   with kernel equal to B.

Goal: Find a map $F : M \to X$ such that the composition

$$\mathbf{C} \otimes TM \xrightarrow{F_*} \mathbf{C} \otimes TX \to T_f^{1,0}$$

is

1. surjective,
2. injective when restricted to TM,
3. has kernel homotopic to B.