CR Structures on Open Manifolds

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Theorem (with P. Landweber)

If

\[ H_p(M^{2n+k}; \mathbb{Z}) = 0 \text{ for } p \geq n + k + 1 \]

then every smooth almost CR structure of codimension k on M is homotopic to a \( C^\omega \) CR structure of codimension k. In particular, every \( C^\infty \) CR structure may be deformed to a \( C^\omega \) CR structure.
**Question 1** Is every five dimensional strictly pseudoconvex CR structure locally realizable?

Every strictly pseudoconvex CR structure on a compact manifold of dimension five or higher is locally realizable. (Boutet de Monvel, 1975)

In particular, does every strictly pseudoconvex CR structure in the neighborhood of a point of $S^5$ extend to all of $S^5$ as a strictly pseudoconvex structure?
Definition
A system of vector fields $L_1, \ldots, L_n$ is aberrant on a domain $\Omega$ if at every point of $\Omega$ the only germs $f$ satisfying

$$L_jf = 0 \text{ for } j = 1, \ldots, n$$

are the constants.

Jacobowitz-Treves (1983) claimed there exist aberrant CR structures of Levi signature $(n - 1, 1)$. Further, we claimed that the aberrant structures are dense.
Theorem

For each CR structure $L$ of signature $(n - 1, 1)$ and each point of $\Omega$ there exists a CR structure $L'$ agreeing with it to infinite order at that point and such that each germ satisfying

$$L'_j f = 0 \text{ for } j = 1, \ldots, n$$

must also satisfy $df = 0$ at the point.

Question 2: Can every CR structure of a given signature in a neighborhood of the origin in $R^{2n+1}$ be extended to a CR structure on all of $R^{2n+1}$ of the same signature?
Question 3: Can every smooth CR structure on a manifold be perturbed to become real analytic?

Yes, if the manifold satisfies the homology conditions of the Theorem.
If

\[ H_p(M^{2n+k}; \mathbb{Z}) = 0 \text{ for } p \geq n + k + 1 \]

then there exists

1. An open covering \( M = \bigcup_j O_j, \quad j \in A \),
2. continuous maps \( f_j : O_j \rightarrow \mathbb{C}^{n+k} \),
3. local biholomorphisms \( \gamma_{ij} \) of \( \mathbb{C}^{n+k} \) defined for each pair \((i, j)\) such that \( O_i \cap O_j \neq \emptyset \) satisfying:
   \[ \gamma_{ik} = \gamma_{ij} \circ \gamma_{jk} \] at all points where both sides are defined, and
   \[ f_i = \gamma_{ij} \circ f_j \text{ on } O_i \cap O_j. \]
\[ \mathcal{B}_{n,k} \times \text{BGL}(n) \xrightarrow{\text{cl}(\nu_n, k) \times \text{B(id)}} \text{BGL}(n + k) \times \text{BGL}(n). \]
Lemma

Under the homological conditions on $M^{2n+k}$ there exists a manifold $X$ of real dimension $4n + 3k$ and an embedding $\iota : M \to X$ such that

1. $X$ is a fiber bundle over $M$ with complex structure on the fibers.
2. $X$ admits a foliation $\mathcal{F}^{2n+k}$ transverse to the fibers.
3. There is a surjective bundle map

$$\mathbb{C} \otimes TM \to \iota^* T_{f}^{1,0}$$

with kernel equal to $B$.

Goal: Find a map $F : M \to X$ such that the composition

$$\mathbb{C} \otimes TM \xrightarrow{F^*} \mathbb{C} \otimes TX \to T_{f}^{1,0}$$

is

1. surjective,
2. injective when restricted to $TM$,
3. has kernel homotopic to $B$. 