Left Invariant CR Structures on S^3

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Outline

- CR structures on M^3
- Pseudo-hermitian structures on M^3
  - Curvature and torsion
- $S^3 = SU(2)$
- Left-invariant CR and pseudo-hermitian structures on S^3
  - Classification results
- Geodesics in sub-Riemannian geometry

Motivation

There are many results about pseudo-hermitian structures that are torsion free:

1. Isoperimetric inequalities (e.g., Chanillo and Yang, 2009)
2. Sasakian geometry and physics

Wanted simple examples of pseudo-hermitian structures with torsion.

Opportunity to work in pseudo-hermitian and subriemannian geometries.

CR Structures

A **CR structure** on M^3 is a two-plane distribution $H \subset TM$ and a complex structure on each fiber.

$$J : H \to H \text{ with } J^2 = -I.$$

We denote this structure by (M, H, J). It is often useful to extend J by complex linearity to a map

$$J : \mathbf{C} \otimes \mathbf{H} \to \mathbf{C} \otimes \mathbf{H}.$$

Then J is completely determined by the eigenspace corresponding to the eigenvalue i (or to the eigenvalue -i). So a CR structure is just as well given by $B \subset \mathbf{C} \otimes H$,

$$B \cap \overline{B} = \{0\}.$$

It is useful to work with the dual formulation. Let $\theta^{\perp} = H$. Assume

$$\theta \wedge d\theta \neq 0.$$

Strict pseudoconvexity

There exists some θ^1 such that

1. $d\theta = i\theta^1 \wedge \overline{\theta^1}$ (or $d(-\theta) = i\theta^1 \wedge \overline{\theta^1}$)
2. $X \in H \implies \theta^1(X + iJX) = 0$. (Equivalently, $J\theta^1 = i\theta^1$)

(θ, θ^1) is called a CR coframe.

The forms θ and θ^1 are not unique. For example

$$\begin{aligned}\tilde{\theta} &= r\theta \\ \tilde{\theta^1} &= \alpha\theta^1\end{aligned}$$

with constants r real and α complex, $|\alpha|^2 = r > 0$.
A **pseudo-hermitian structure** is a CR coframe (θ, θ^1) with θ fixed.

$$\begin{aligned}\tilde{\theta} &= \theta \\ \tilde{\theta^1} &= \alpha\theta^1\end{aligned}$$

with $|\alpha| = 1$.

The Standard Structures

The **standard CR structure** on the three sphere S^3 is the one it inherits as a submanifold of $\mathbf{C}^2$.

$$H = TS^3 \cap JTS^3.$$

H is called the **standard contact distribution**.
Choosing $\theta_0 = -i(\overline{z}dz + \overline{w}dw)$ give the **standard pseudo-hermitian structure** .

The natural choice for a coframe for these structures is

$$\{\theta_0, \theta_0^1\}$$

with

$$\begin{aligned}\theta_0 &= -i(\overline{z}dz + \overline{w}dw) \\ \theta_0^1 &= wdz - zdw\end{aligned}$$

where these forms are restricted to S^3.
We have

$$\begin{aligned}d\theta_0 &= i\theta_0^1 \wedge \overline{\theta_0^1} \\ d\theta_0^1 &= \theta_0^1 \wedge \omega \\ \omega &= -2i\theta_0.\end{aligned}$$

Given two CR structures (M, H, J) and $(M, \tilde{H}, \tilde{J})$ a diffeomorphism $F : M \to M$ is a **CR diffeomorphism** if it preserves the two-plane distribution and the J-operator. That is

$$F_* \circ J = \tilde{J} \circ F_*.$$

In terms of choices of coframes we are requiring

$$F^* \tilde{\theta} = s\theta$$

$$F^* \tilde{\theta^1} = \gamma\theta^1 + \delta\theta$$

with s real, γ and δ complex $s \neq 0$, and $\gamma \neq 0$.

Given two pseudo-hermitian structures, say $\{\theta, \theta^1\}$ and $\{\theta, \tilde{\theta^1}\}$ and a diffeomorphism $F : M^3 \to M^3$, we say that the two pseudo-hermitian structures are equivalent, and that F is a **pseudo-hermitian diffeomorphism** if

$$F^*(\theta) = \theta$$

and

$$F^*(\tilde{\theta^1}) = \gamma\theta^1 + \delta\theta.$$

($\gamma \neq 0$)

The Webster Connection

Theorem

Let (θ, θ^1) be a pseudo-hermitian coframe. There exist unique functions R, A, and V, and an unique one-form ω, so that

$$\begin{aligned} d\theta &= i\theta^1 \wedge \overline{\theta^1} \\ d\theta^1 &= \theta^1 \wedge \omega + A\theta \wedge \overline{\theta^1} \\ \omega &= -\overline{\omega} \\ d\omega &= R\theta^1 \wedge \overline{\theta^1} + 2i\Im(V\overline{\theta^1}) \wedge \theta. \end{aligned}$$

Further, if θ^1 is replaced by $\boldsymbol{\theta}^{\mathbf{1}} = \lambda\theta^1$, $|\lambda| = 1$, then

$$\mathbf{R} = R, \quad \mathbf{A} = \lambda^2 A, \quad \mathbf{V} = \lambda V, \quad \boldsymbol{\omega} = \omega - \lambda^{-1} d\lambda.$$

The group structure

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

$$SU(2) \leftrightarrow S^3$$

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \leftrightarrow (\alpha, \beta) \in C^2$$

Let $\alpha = a + ib$ and $\beta = c + id$.
Now $SU(2)$ acts on R^4.
Starting with the vectors

$$(0,1,0,0), \quad (0,0,1,0), \quad \text{and } (0,0,0,1)$$

tangent to S^3 at $(1,0,0,0)$, we translate them using $SU(2)$ to obtain the vector fields at the point (a,b,c,d)

$$\begin{aligned} L_1 &= (-b, a, -d, c)) \\ L_2 &= (-c, d, a, -b) \\ L_3 &= (-d, -c, b, a). \end{aligned}$$

Contact structures

Each of the three 2-planes spanned by $\{L_j, L_k\}$ is contact. For example,

$$H = \{L_2, L_3\}$$

is the standard contact structure. Thus the standard contact structure in left-invariant.

For any other left-invariant distribution we can choose a basis

$$\begin{aligned} U &= L_1 + uL_3 \\ V &= L_2 + vL_3 \end{aligned}$$

with real constants u and v.

Lemma

Each left-invariant 2-plane distribution on S^3 is a contact structure.

Proof

We have

$$\frac{1}{2}[U, V] = uL_1 + vL_2 - L_3.$$

Assume

$$[U, V] = xU + yV$$

Then

$$x = 2u, \quad y = 2v, \text{ and } xu + yv = -2$$

gives the contradiction

$$2u^2 + 2v^2 = -2.$$

Lemma

If $\mathcal{D}$ is a left-invariant 2-plane distribution on S^3 then there is some $\Phi : S^3 \to S^3$ such that the induced map

$$\Phi_* TS^3 \to TS^3$$

takes $\mathcal{D}$ to the standard contact distribution.

Restrict CR and pseudo-hermitian structures to have the standard distribution.

Left-Invariant Structures

(Almost) any complex structure on H at a given point is given by

$$\theta^1 = \theta_0^1 + \mu\overline{\theta_0^1}, \quad \mu \neq \pm 1.$$

As μ varies we obtain all the CR structures with the given contact distribution, except for the conjugate of the standard CR structure which appears as the limit as $\mu \to \infty$.
Pseudo-hermitian coframe

$$\begin{aligned} \theta &= \theta_0 \\ \theta^1 &= \lambda(\theta_0^1 + \mu\overline{\theta_0^1}) \end{aligned}$$

with

$$|\lambda|^2(1 - |\mu|^2) = 1.$$

We have

$$d\theta^1 = \theta^1 \wedge \left(-2i\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right)\right)\theta - \frac{4i\mu}{1-|\mu|^2}\theta \wedge \overline{\theta^1}.$$

Webster connection form

$$\omega = -2i\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right)\theta_0.$$

Torsion

$$A = -\frac{4i\mu}{1-|\mu|^2}$$

Curvature

$$R = 2\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right).$$

Let $\mathcal{S}$ denote the set of equivalence classes of left invariant pseudo-hermitian structures corresponding to the standard contact structure.

Theorem

The map

$$\{\mu \in C,\ |\mu| < 1\} \to \mathcal{S}$$

is surjective with fiber given by $|\mu|$.

Remark

The same result and proof hold for $|\mu| > 1$ *and* θ *replaced by* $-\theta$.

Proof

Assume $|\mu| = |\mu'|$. Let $F(z,w) = (\zeta z, w) = (\tilde{z}, \tilde{w})$ for $\zeta \in \mathbf{C}$ and $|\zeta| = 1$.

$$F^*(\theta|_{(\tilde{z},\tilde{w})}) = \theta|_{(z,w)}$$

and

$$\begin{aligned} F^*(\tilde{\theta^1}|_{(\tilde{z},\tilde{w})}) &= F^*(\tilde{\lambda}(\theta_0^1 + \tilde{\mu}\overline{\theta_0^1})) \\ &= \tilde{\lambda}(\zeta(wdz - zdw) + \tilde{\mu}\overline{\zeta}(\overline{w}d\overline{z} - \overline{z}d\overline{w})) \\ &= k\theta^1|_{(z,w)} \end{aligned}$$

Provided we can find some ζ of norm one such that $\tilde{\mu}\overline{\zeta}/\zeta = \mu$. That is, provided

$$|\tilde{\mu}| = |\mu|$$

Remark When $\mu = \tilde{\mu}$, we can take $\zeta = \pm 1$ and we have two CR diffeomorphisms leaving $(0,1)$ fixed.
This illustrates a general theorem:
A spc CR structure on M^3 with nonzero "CR curvature" admits at most two CR diffeomorphisms leaving a given point fixed.
If $\mu = 0$ the dimension of the isotropy group of a point is 5.

Conversely, we start with a pseudo-hermitian diffeomorphism F and show that $|\mu| = |\tilde{\mu}|$.

$$\begin{aligned} d\theta^1 &= \theta^1 \wedge \omega + A\theta \wedge \overline{\theta^1} \\ d\tilde{\theta^1} &= \tilde{\theta^1} \wedge \tilde{\omega} + \tilde{A}\theta \wedge \overline{\tilde{\theta^1}} \end{aligned}$$

together with $F^*(\tilde{\theta^1}) = \alpha\theta^1$ to derive

$$(d\alpha) \wedge \theta^1 + \alpha(\theta^1 \wedge \omega + A\theta \wedge \overline{\theta^1}) = \alpha\theta^1 \wedge \tilde{\omega} + \overline{\alpha}\tilde{A}\theta \wedge \overline{\theta^1}.$$

Wedge this with θ^1 and obtain

$$\alpha A = \overline{\alpha}\tilde{A}.$$

Use

$$A = -\frac{4i\mu}{1-|\mu|^2}$$

to conclude that $|\mu| = |\tilde{\mu}|$.

CR structural equations

Let ϕ and ϕ_1 be one-forms with ϕ real giving the CR structure:

1. $\phi^{\perp} = H$,
2. $J\phi_1 = i\phi_1$,
3. $d\phi = i\phi_1\overline{\phi_1}$.

Theorem

There exist unique one-forms ϕ_2, ϕ_3, ϕ_4 and unique functions $R(x)$ and $S(x)$ such that

1. *ϕ_2 is imaginary and ϕ_4 is real,*
2. $d\phi_1 = -\phi_1\phi_2 - \phi\phi_3$,
3. $d\phi_2 = 2i\phi_1\overline{\phi_3} + i\overline{\phi_1}\phi_3 - \phi\phi_4$,
4. $d\phi_3 = -\phi_1\phi_4 - \overline{\phi_2}\phi_3 - R\phi\overline{\phi_1}$,
5. $d\phi_4 = i\phi_3\overline{\phi_3} + (S\phi_1 + \overline{S\phi_1})\phi$.

4 If we replace ϕ by $\psi = |\nu|^2\phi$ and ϕ_1 by $\psi_1 = \nu\phi_1$ with a constant ν then the forms

$$\psi_2 = \phi_2, \quad \psi_3 = \frac{1}{\overline{\nu}}\phi_3, \quad \psi_4 = \frac{1}{|\nu|^2}\phi_4$$

satisfy the equations in the Theorem with R and S replaced by

$$R = \frac{R}{|\nu|^2\overline{\nu}^2} \text{ and } S = \frac{S}{|\nu|^2\nu}.$$

R is a relative invariant.

$R(p) \neq 0$ implies that (M, H, J) is nonumbilic at p.

Corollary

A left invariant CR structure on S^3 with $\mu \neq 0$ has no umbilic points.

We want to choose a multiple of ϕ and a corresponding multiple of ϕ_1 so that $R(x) \equiv 1$.

Corollary

If $R(p) \neq 0$, there are precisely two choices of (ϕ, ϕ_1) such that in a neighborhood of p

1. *(ϕ, ϕ_1) give the CR structure,*
2. *$d\phi = i\phi_1\overline{\phi_1}$, and*
3. *$R \equiv 1$.*

If we denote one choice by (ω, ω_1), then the other choice is $(\omega, -\omega_1)$.We set $\phi = \omega$ and $\phi_1 = \omega_1$ and apply the theorem to obtain ϕ_2, ϕ_3, and ϕ_4.

$$\phi_2' = \phi_2, \quad \phi_3' = -\phi_3, \text{ and } \phi_4' = \phi_4.$$

Theorem

If F is a CR diffeomorphism between left-invariant CR structures characterized by μ *and* $\tilde{\mu}$ *and with the standard contact distribution then either* $|\mu| = |\tilde{\mu}|$ *or* $|\mu| = 1/|\tilde{\mu}|$.

Geodesics

Local coordinates for T^*M

$$(x, \xi) \to (x, \xi_j dx_j)$$

Global symplectic form

$$\omega_S = -d(\xi_j dx_j)$$

For pseudo-hermitian structure

$$-\omega_S = d(\zeta\theta^1 + \overline{\zeta}\overline{\theta^1} + \eta\theta)$$

Hamiltonian

$$H = |\zeta|^2$$

The Hamiltonian vector field on T^*M

$$X_H = \overline{\zeta} Z + \zeta \overline{Z} + B\partial_\zeta + \overline{B}\partial_{\overline{\zeta}} + C\partial_\eta$$

with

$$\begin{aligned} B &= i\eta\zeta + \zeta\omega(X) \\ C &= \zeta^2 A + \overline{\zeta}^2 \overline{A}. \end{aligned}$$

$\gamma(t)$ is the projection of an integral curve of X_H to M.

Geodesics in terms of the connection

Let $(T, Z, \overline{Z})$ be dual to a pseudo-hermitian coframe $(\theta, \theta^1, \overline{\theta^1})$. A connection is defined by

$$\begin{aligned} \nabla Z &= \omega \otimes Z \\ \nabla \overline{Z} &= -\omega \otimes \overline{Z} \\ \nabla T &= 0. \end{aligned}$$

Theorem

The curve $\gamma(t)$ is Legendrian and its tangent $X(t)$ satisfies

$$\nabla_X X = aJX \text{ and } Xa = < Tor(X, T), X > .$$

The geodesic equations on T^*M

$$
\begin{aligned}
\gamma'(t) &= \overline{\zeta(t)}Z(\gamma(t)) + \zeta(t)\overline{Z(\gamma(t))} \\
\zeta'(t) &= i\eta\zeta + \omega(\gamma')\zeta \\
\eta'(t) &= 2\Re(\zeta^2 A).
\end{aligned}
$$

$|\zeta(t)|$ is a constant.
$\zeta(t)$ gives the direction of $\gamma(t)$ in $H_{\gamma(t)}$.
$\eta(t)$ is related to the curvature of $\gamma(t)$.

Restrict to left-invariant pseudo-hermitian structures

Theorem

Let $\gamma(t)$ be a geodesic for a left invariant structure and let $P = \gamma(t_0)$. Let $\alpha(t)$ be the projection of $\gamma(t)$ into the horizontal complex line at P. Let ζ and η be the associated functions. Then the curvature of $\alpha(t)$ at $t = t_0$ is equal to

$$\frac{|\eta(t_0)|}{|\lambda(\overline{\zeta(t_0)} - \mu\zeta(t_0)|^3}.$$

The well-known result for the hyperquadric Q asserting that the projected curves $\alpha(t)$ are circles does not hold for S^3.

Restrict to the standard left-invariant structure

$$A = 0 \text{ and so } \eta \text{ is a constant.}$$

Theorem

Each geodesic (as a space curve) has constant curvature given by

$$\kappa = \sqrt{1 + \eta^2}.$$

Theorem

The geodesic satisfying

$$\gamma(0) = (0,1) \qquad (1)$$
$$\gamma'(0) = (e^{i\phi},0) \qquad (2)$$
$$\gamma''(0) = (i\eta e^{-\phi},-1) \qquad (3)$$

is periodic if and only if

$$\frac{\eta}{\sqrt{\eta^2+4}} \qquad (4)$$

is rational.

THANK YOU