Left Invariant CR Structures on $SL(2, \mathbf{R})$

Howard Jacobowitz
Rutgers University Camden
August 2017

CR Structures

$$L = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + i(x + iy)\frac{\partial}{\partial t}$$

A CR Structure on a 3-dimensional manifold is the set of multiples of a complex vector field L_1 where L_1 and $\overline{L_1}$ are everywhere independent.

Analysis

Lewy (1957) For most functions $f(x, y, t)$ the equation

$$Lu = f$$

has no solutions in any open set of $\mathbf{R^3}$.

Geometry
Cartan (1932) Pseudo-conformal geometry
$\Re L$ and $\Im L$ define a 2-plane distribution in TM^3.
The map

$$\Re L \to \Im L \text{ and } \Im L \to -\Re L$$

defines an operator J, $J^2 = -1$.
Equivalence problem:

A CR Structure on M^3 is a complex line sub-bundle $V \subset \mathbf{C} \otimes \mathbf{TM}$ such that $V \cap \overline{V} = \{0\}$.

$$V_1 \sim V_2$$

if there exists a diffeomorphism $\phi : M \to M$ with

$$\phi_* V_1 = V_2.$$

Example

$$M = \text{ Lie group } = G = SL(2, \mathbf{R})$$

(or $SU(2, \mathbf{R})$ from 2 years ago).

$$v|_e \in \mathbf{C} \otimes TM|_e, \quad \Re v|_e \text{ and } \Im v|_e \text{ independent}$$

G acts on G by $L_g(h) = gh$ and acts on $\mathbf{C} \otimes TG$ by L_{g*}.
$v|_e$ extends to a CR vector field by

$$v_g = L_{g*} v|_e.$$

This vector field is left-invariant:

$$v|_{gh} = L_{g*} v|_h$$

for all g and h in G.
Note: If v is left-invariant then so is $R_{g*} v$ for all $g \in G$.

Goal: Classify the left-invariant CR structures on $SL(2, \mathbf{R})$.
To explain, start with the relation of $SL(2, \mathbf{R})$ to complex variables:
Recall the hyperbolic (or Poincaré) geometry of the upper half-plane.

The distance between two points in the upper half-plane is

$$\cosh d(z,w) = 1 + \frac{|z-w|^2}{2\Im z \Im w}$$

and corresponds to the displacement element

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The holomorphic mappings of the upper half-plane to itself are given by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R})$$

acting by

$$g(z) = \frac{az+b}{cz+d}$$

and these are isometries of the upper half-plane

$$d(g(z),g(w)) = d(z,w).$$

Further, if

$$d(z_1, w_1) = d(z_2, w_2)$$

then there exists a unique $g \in SL(2, \mathbf{R})$ with

$$\begin{aligned} g(z_1) &= g(z_2) \\ g(w_1) &= g(w_2). \end{aligned}$$

The Main Lemma

$$G = SL(2, \mathbf{R}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } ad - bc = 1.$$

$$TG\big|_e = sl(2, \mathbf{R}) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

$$P_2 = \{p(z) = \lambda z^2 + \mu z + \nu\}.$$

Main Lemma There exists a 'natural' map

$$q : sl(2, \mathbf{C}) \to P_2.$$

Associate to a left-invariant CR structure $V \subset \mathbf{C} \otimes TG$ the pair of roots

$$\zeta_1 \text{ and } \zeta_2$$

of $q(L) = 0$, where L is any non-zero vector in $V|_e$. Sometimes write $q(V)$ in place of $q(L)$.
For a CR structure, neither root is real.
Divide the CR structures into two classes:
E={ Both roots are in the same half-space }
and
H={ The roots are in opposite half-spaces }.
Define
$d(V) = d(\zeta_1, \zeta_2)$ for $V \in E$
and
$d(V) = d(\zeta_1, \overline{\zeta_2})$ for $V \in H$.

Assume V_1 and V_2 are in the same class.

Theorem (Jacobowitz)

1. *If* $d(V_1) = d(V_2)$ *then* $V_1 \sim V_2$.
2. *If* $d(V_1) \neq d(V_2)$ *then* $V_1 \nsim V_2$.

Assume V_1 and V_2 are in the same class.

Theorem (Jacobowitz)

1. *If* $d(V_1) = d(V_2)$ *then* $V_1 \sim V_2$.
2. *If* $d(V_1) \neq d(V_2)$ *then* $V_1 \nsim V_2$.

Assume V_1 and V_2 are in the same class.

Theorem (Bor, Jacobowitz)

1. *If* $d(V_1) = d(V_2)$ *then* $V_1 \sim V_2$.
2. *If* $d(V_1) \neq d(V_2)$ *then* $V_1 \nsim V_2$.

Notation

Let G be a Lie group.

$$\begin{aligned} \mathcal{G} &= TG|_e \\ &= \text{Lie algebra of G} \\ &= \{ \text{ left-invariant vector fields on } G\}. \end{aligned}$$

A representation of a group G on a vector space V is a group homomorphism

$$G \to GL(V).$$

There is a natural representation of G on $\mathcal{G}$

$$Ad : G \to GL(\mathcal{G})$$

and on $\mathbf{C} \otimes \mathcal{G}$. For $G = SL(2, \mathbf{R})$ the representation is just

$$Ad_g(M) = gMg^{-1}, \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}.$$

There is a natural action of G on a function space whenever G acts on the domain

$$g(f)(x) = f(g^{-1}x).$$

For $G = SL(2, \mathbf{R})$ acting as linear fractional transformation on $\mathbf{C}$ and $V = P_2$

$$p(z) \to p(g^{-1}(z)).$$

These actions of $SL(2, \mathbf{R})$ on $\mathcal{G}$ and P_2 are **irreducible**.

$$sl(2) \xrightarrow{g} sl(2)$$

$$P_2 \xrightarrow{g} P_2$$

$$\begin{array}{ccc} sl(2) & \xrightarrow{\;g\;} & sl(2) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q} \\ P_2 & \xrightarrow{\;g\;} & P_2 \end{array}$$

$$\begin{array}{ccc} sl(2) & \xrightarrow{g} & sl(2) \\ \downarrow q & & \downarrow q \\ P_2 & \xrightarrow{g} & P_2 \end{array}$$

So

$$q(g(M)) = g(q(M)).$$

$$\begin{array}{ccc} sl(2) & \xrightarrow{g} & sl(2) \\ \downarrow q & & \downarrow q \\ P_2 & \xrightarrow{g} & P_2 \end{array}$$

So

$$q(g(M)) = g(q(M)).$$

That is,

$$q(gMg^{-1}) = q(M) \circ g^{-1}.$$

From

$$q(gMg^{-1}) = q(M) \circ g^{-1}.$$

we see that if ζ_1 and ζ_2 are the roots of $q(M)$ then, $g(\zeta_1)$ and $g(\zeta_2)$ are the roots of gMg^{-1}.
The roots determine M up to multiples because q determines M:

$$q\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = cz^2 - 2az - b.$$

Let V have the roots (ζ_1, ζ_2) and W have the roots (η_1, η_2). If

$$d(\zeta_1, \zeta_2) = d(\eta_1, \eta_2)$$

then there exists $g \in SL(2, \mathbf{R})$ such that

$$g(\zeta_1) = \eta_1 \text{ and } g(\zeta_2) = \eta_2.$$

So V and gWg^{-1} have the roots $(g^{-1}\eta_1, g^{-1}\eta_2)$. Thus

$$V = gWg^{-1} = R_{g^{-1}} * W.$$

This shows that if the Poincaré distances are the same, then the CR structures are equivalent.

Showing that the structures are inequivalent when the distances are different uses Cartan's method of moving frames.

1. Find explicit representations in each class for every Poincaré distance.
2. Compute a Cartan invariant μ for each representative.
3. Show that $\mu(V) = \mu(W) \implies d(V) = d(W)$.

Thus,

$$d(V) \neq d(W) \implies V \nsim W.$$

Step one

Let $0 < \tau \leq 1/4$ and set

$$L_\tau = \begin{pmatrix} -2i\tau & \tau \\ 1 & 2i\tau \end{pmatrix} = -2i\tau\ \mathbf{h} + \tau\ \mathbf{e} + \ \mathbf{f}.$$

Then

1. The roots are both in the lower half-plane.
2. $$\cosh d = 1 + \frac{1 - 4\tau}{2\tau}.$$
3. The map $\tau \to d$ is bijective.
4. There is a multiple root when $\tau = 1/4$.

It is useful to work with the dual formulation. Let $\theta^{\perp} = \{\Re V\}$.
$V \in E$ (or $V \in H$) implies

$$\theta \wedge d\theta \neq 0.$$

There exists some θ^1 such that

1. $d\theta = i\theta^1 \wedge \overline{\theta^1}$ (or $d(-\theta) = i\theta^1 \wedge \overline{\theta^1}$)
2. $X \in V \implies \theta^1(X) = 0.$

(θ, θ^1) is called a CR coframe.

Second Step

The forms θ and θ^1 are not unique. For example

$$\begin{aligned} \tilde{\theta} &= r\theta \\ \tilde{\theta^1} &= \alpha\theta^1 \end{aligned}$$

with constants r real and α complex, $|\alpha|^2 = r > 0$. How can we make a canonical choice of these forms?

Second Step

Theorem

(Cartan) Given one-forms θ and θ^1 with θ real, $\theta \wedge \theta^1 \wedge \overline{\theta^1} \neq 0$ and $d\theta = i\theta^1 \wedge \overline{\theta^1}$, there exist unique forms θ^2, θ^3, and θ^4 such that θ^2 is imaginary and θ^4 is real and unique functions R and S such that

$$\begin{aligned}
d\theta &= i\theta^1 \wedge \overline{\theta^1} \\
d\theta^1 &= -\theta^1 \wedge \theta^2 - \theta \wedge \theta^3 \\
d\theta^2 &= 2i\theta^1 \wedge \overline{\theta^3} + i\overline{\theta^1} \wedge \theta^3 - \theta \wedge \theta^4 \\
d\theta^3 &= -\theta^1 \wedge \theta^4 - \overline{\theta^2} \wedge \theta^3 - R\theta \wedge \overline{\theta^1} \\
d\theta^4 &= i\theta^3 \wedge \overline{\theta^3} + (S\theta^1 + \overline{S}\overline{\theta^1}) \wedge \theta.
\end{aligned}$$

Theorem

The CR structure defined by θ and θ^1 is locally equivalent to the standard CR structure on S^3 if and only if $R = 0$. Further, in this case $S = 0$.

Second Step

Theorem

If $R \neq 0$ there exists some function λ such that the system associated to $\omega = |\lambda|^2\theta$ and $\omega^1 = \lambda\theta^1$ has $R' = 1$, where R' is the value of R computed starting with ω and ω^1. The only other function which achieves this is $-\lambda$.

Thus, provided $R \neq 0$, there exists a unique (except for a sign ambiguity) choice of the CR frame.
We apply this to L_τ.

$$\begin{aligned} \theta &= \theta_e - \tau\theta_f \\ \theta^1 &= i\theta_h - \tau\theta_f - \theta_e. \end{aligned}$$

Second Step

Cartan's equations are satisfied when

$$\begin{aligned}
\theta^2 &= \frac{3}{4}\Lambda\theta \\
\theta^3 &= -\frac{1}{4}\Lambda\theta^1 - U\overline{\theta^1} \\
\theta^4 &= (\frac{1}{16}|\Lambda|^2 - |U|^2)\theta \\
R &= -\frac{3}{2}U\Lambda = \frac{3}{2}(\frac{1}{16\tau^2} - 1) \\
S &= 0,
\end{aligned}$$

where

$$\begin{aligned}
U &= i(\frac{1}{4\tau} - 1) \\
\Lambda &= i(\frac{1}{4\tau} + 1).
\end{aligned}$$

Second Step

$$\lambda = \pm |R|^{1/2}$$

gives us the unique (up to a possible minus sign) forms that achieve $R' = 1$. Then

$$\begin{aligned} \omega^2 &= \frac{1}{2}\Lambda|\Lambda U|^{-1/2}\omega \\ &= \mu(\tau)\omega. \end{aligned}$$

Third Step

In this way we have assigned to each L_τ, $\tau \neq 1/4$ the function $\mu(\tau)$. We now show that if the CR structures L_τ and $L_{\tilde{\tau}}$ are locally equivalent, then

$$\mu(\tau) = \mu(\tilde{\tau}).$$

So consider two values τ and $\tilde{\tau}$. Let Φ be a CR equivalence

$$\Phi_* V_\tau = V_{\tilde{\tau}}.$$

By the uniqueness of the forms we have

$$\Phi^*(\tilde{\omega}) = \omega \text{ and } \Phi^*(\tilde{\omega^1}) = \pm\omega^1$$

and similar equations for the other forms.

In particular

$$\Phi^*(\tilde{\omega^2}) = \omega^2$$

and so

$$\Phi^*(\mu(\tilde{\tau})\tilde{\omega}) = \mu(\tau)\omega.$$

That is,

$$\mu(\tau) = \mu(\tilde{\tau})$$

which in turn implies

$$\tau = \tilde{\tau}.$$

Let V and $\tilde{V}$ be in class E. We want to show $V \sim \tilde{V}$ implies $d(V) = d(\tilde{V})$. There exist τ and $\tilde{\tau}$ with

$$d(V) = d(V_\tau) \text{ and } d(\tilde{V}) = d(V_{\tilde{\tau}}). \quad (1)$$

Thus also

$$V \sim V_\tau \text{ and } \tilde{V} \sim V_{\tilde{\tau}}.$$

Now, if we assume $V \sim \tilde{V}$ we also have $V_\tau \sim V_{\tilde{\tau}}$ which in turn implies $\tau = \tilde{\tau}$ and hence

$$d(V_\tau) = d(V_{\tilde{\tau}}). \quad (2)$$

Thus from (1) and (2) we have

$$d(V) = d(\tilde{V}).$$

The CR structures in E and in H are inequivalent, with one exception.
The locally spherical CR structure has representatives in each class.
The Poincaré distances are 0 and $cosh^{-1}3$.

THANK YOU