GEODESICS OF LEFT-INVARIANT PSEUDO-HERMITIAN STRUCTURES ON THE THREE DIMENSIONAL SPHERE

HOWARD JACOBOWITZ

ABSTRACT. The three-dimensional case of the structure equations of Webster are summarized and then used to determine the sub-Riemannian geodesics of the left-invariant CR structures on the group $SU(2,\mathbf{R}) = S^3$. These geodesics are studied in detail for the standard left-invariant CR structure.

In memory of Nicholas Hanges, a friend and colleague

1. CR and pseudo-hermitian structures

A **CR structure** on a three dimensional manifold M is a two-plane distribution $H \subset TM$ and a fiber preserving anti-involution $J : H \to H$.

An equivalent definition of a CR structure on a 3 dimensional manifold may be given in terms of a complex line bundle: A CR structures on M is a line bundle $B \subset C \otimes TM$ with the property that $B \cap \overline{B}$ contains only the zero section. Then

$$H = \{\Re Z \ : \ Z \in B\}$$

is of rank 2 and J is defined on $\mathbf{C} \otimes H = B \oplus \overline{B}$ by setting

$$J(Z) = iZ \text{ if } Z \in B$$

and

$$J(Z) = -iZ \text{ if } Z \in \overline{B}.$$

So for $X + iY \in B$

$$JX = Y \text{ and } JY = -X.$$

This definition makes it clear that each $M^3 \subset \mathbf{C}^2$ has an induced CR structure. Just take

$$B = C \otimes TM \cap T^{1,0}(\mathbf{C}^2).$$

There is a third definition. Let θ be a real one form and θ^1 a complex one form with $\theta^1 \wedge \overline{\theta^1} \neq 0$. Define $B \subset C \otimes TM$ by

$$\theta(B) = 0 \text{ and } \overline{\theta^1}(B) = 0. \tag{1.1}$$

Or, start with B and use these equations to define the forms. Note that the forms θ and θ^1 defined this way are not unique. All other choices satisfying (1.1) are given by

$$\begin{aligned} \tilde{\theta} &= r\theta \\ \tilde{\theta^1} &= \alpha\theta^1 + \beta\theta \end{aligned} \tag{1.2}$$

with r real, α and β complex, $r \neq 0$ and $\alpha \neq 0$.

Thus to each CR structure there corresponds at each point a five dimensional choice of coframes.

We only consider CR structures that satisfy the addition property that the line bundles B, $\overline{B}$, and $[B, \overline{B}]$ are independent. These are the strictly pseudo-convex CR structures. This subject has its roots in an observation of Poincaré [5] and the seminal papers of Cartan [2] and [3]. Much of the modern work on the geometry of these structures (in odd dimensions greater than or equal to three) can be traced to [6].

Example: The standard CR structure on the three sphere S^3 is the one it inherits as a submanifold of C^2. The CR bundle is the complex line bundle spanned by the Lewy operator, but now we change notation and set

$$Z_0 = \overline{w}\partial_z - \overline{z}\partial_w. \tag{1.3}$$

So H coincides with the standard contact distribution. That is, H is the span of the real and imaginary parts of Z_0 and $J : H \to H$ is defined by setting $JZ_0 = iZ_0$. So a coframe may be given as

$$\{\theta_0, \theta_0^1\}$$

with

$$\theta_0 = -i(\overline{z}dz + \overline{w}dw) \tag{1.4}$$

$$\theta_0^1 = wdz - zdw \tag{1.5}$$

The frame dual to $\{\theta, \theta_0^1, \overline{\theta_0^1}\}$ is $\{T, Z_0, \overline{Z_0}\}$ where

$$T = i(z\partial_z + w\partial_w) - i(\overline{z}\partial_{\overline{z}} + \overline{w}\partial_{\overline{w}}). \tag{1.6}$$

Note that $|T| = 1$ and $|Z_0| = 1/\sqrt{2}$. (Later we renormalize the Euclidean metric to have $|Z_0| = 1$.)

Definition 1. *Let θ define a contact structure. The unique vector field T satisfying the interior multiplication (contraction) conditions*

$$T \lrcorner \theta = 1, \qquad T \lrcorner d\theta = 0$$

is called the Reeb vector field.

The vector field given by (1.6) is the Reeb vector field of the standard contact structure.

We have seen that a CR structure on M^3 is specified by prescribing H and J. If this structure is strictly pseudo-convex and we fix some real θ with $\theta^\perp = H$, then (H, J, θ) is called a **pseudo-hermitian structure**. A pseudo-hermitian **coframe** is by definition given by (θ, θ^1) with θ^1 satisfying

$$\theta^1(X + iJX) = 0$$

for all $X \in H$. If in addition

$$d\theta = i\theta^1 \wedge \overline{\theta^1} \tag{1.7}$$

then we have a **normalized coframe**.

A pseudo-hermitian structure (H, J, θ) is **positively oriented** if it admits a normalized coframe and **negatively oriented** if $(H, J, -\theta)$ admits a normalized coframe. We mainly state our results for the positively oriented structures. The same results hold, mutatis mutandis, for the negatively oriented ones.

Lemma 1.1. *We have*

1. *Precisely one of the pseudo-hermitian structures* (H, J, θ) *and* $(H, J, -\theta)$ *admits a normalized coframe.*
2. *Let* (θ, θ^1) *be a normalized coframe for some pseudo-hermitian structure. Then* $(\theta, \tilde{\theta^1})$ *is a normalized coframe for the same structure if and only if there exists a function* λ, $|\lambda| = 1$, *with*

$$\tilde{\theta^1} = \lambda\theta^1.$$

3. *A pseudo-hermitian structure* (H, J, θ) *is positively oriented if and only if for some coframe*

$$\theta \wedge d\theta = iC\theta \wedge \theta^1 \wedge \overline{\theta^1}$$

with $C > 0$.

2. The Structure Equations

The set of structure equations for a pseudo-hermitian manifold is given by the next theorem. It is the lowest dimensional case of the well-known construction due to Webster [10], see also Tanaka [9]. The objects produced are unique connections, curvatures, and torsions. Note that for a pseudo-hermitian structure the freedom in choosing θ^1 for the normlized pseudo-hermitian coframe is, according to Lemma 1.1, $\theta^{\mathbf{1}} = \lambda\theta^1$ with $|\lambda| = 1$.

Theorem 2.1. *Let* (θ, θ^1) *be a normalized coframe. There exist unique functions* R, A*, and* V*, with* R *real, and an unique one-form* ω*, so that*

$$\begin{aligned} d\theta &= i\theta^1 \wedge \overline{\theta^1} \\ d\theta^1 &= \theta^1 \wedge \omega + A\theta \wedge \overline{\theta^1} \\ \omega &= -\overline{\omega} \\ d\omega &= R\theta^1 \wedge \overline{\theta^1} + 2i\Im(V\overline{\theta^1}) \wedge \theta. \end{aligned}$$

We see that the associated connection is

$$\begin{aligned} \nabla Z &= \omega \otimes Z \\ \nabla \overline{Z} &= -\omega \otimes \overline{Z} \\ \nabla T &= 0. \end{aligned}$$

Recall that for the standard structure we have as in (1.4) and (1.5)

$$\begin{aligned} \theta_0 &= -i(\overline{z}dz + \overline{w}dw) \\ \theta_0^1 &= wdz - zdw \end{aligned}$$

and so

$$d\theta_0 = i\theta_0^1 \wedge \overline{\theta_0^1} \tag{2.1}$$

$$d\theta_0^1 = \theta_0^1 \wedge \omega \tag{2.2}$$

$$\omega = -2i\theta_0. \tag{2.3}$$

So for the standard structure the curvature is given by $R = 2$, the connection by $\omega = -2i\theta$ and the torsion by $A = 0$.

3. The left-invariant structures on S^3

We specialize the Webster construction under the assumption of left-invariance and the standard contact distribution.

Let θ and θ^1 be a coframe of some such pseudo-hermitian structure on S^3. At any specific point we have

$$\begin{aligned} \theta &= k\theta_0 \\ \theta^1 &= \lambda(\theta_0^1 + \mu\overline{\theta_0^1}), \quad |\mu| \neq 1. \end{aligned}$$

Here k is real and λ and μ are complex with k and λ nonzero and θ_0 and θ_0^1 are given by (1.4) and (1.5). Because of the invariance, these numbers are constants and these equations hold at all points of S^3. As μ varies, we obtain all the pseudo-hermitian structures with the given contact distribution, except for the conjugate of the usual CR structure which appears here at the limit as $\mu \to \infty$.

Lemma 3.1. (θ, θ^1) *is a positively oriented pseudo-hermitian structure if and only if either*

(1) $k > 0$ *and* $|\mu| < 1$, *or*
(2) $k < 0$ *and* $|\mu| > 1$.

Proof. It suffices to show this at one point, say $z = 1$ and $w = 0$. At this point

$$\theta = -ikdz, \quad d\theta = ik(dz \wedge d\bar{z} + dw \wedge d\overline{w}),$$
$$\theta_0^1 = -dw, \quad \theta^1 = -dw + \mu d\overline{w}.$$

So

$$\theta \wedge d\theta = k^2 dz \wedge dw \wedge d\overline{w}$$

and

$$i\theta \wedge \theta^1 \wedge \overline{\theta^1} = k(1 - |\mu|^2) dz \wedge dw \wedge d\overline{w}.$$

The result follows from Lemma 1.1. □

For simplicity, we henceforth take $k = 1$. Otherwise, some formulas would need to be slightly modified.

Now we start with

$$\theta = \theta_0$$

and

$$\theta^1 = \lambda(\theta_0^1 + \mu\overline{\theta_0^1}).$$

For a given μ, we normalize the choice of θ^1 by choosing λ so that (1.7) holds. We first treat the case of $|\mu| < 1$.

This leads to the normalized coframe

$$\theta = \theta_0 \tag{3.1}$$
$$\theta^1 = \lambda(\theta_0^1 + \mu\overline{\theta_0^1}) \tag{3.2}$$

with

$$|\lambda|^2(1 - |\mu|^2) = 1. \tag{3.3}$$

The Lewy vector fields are then given by the solutions to

$$\theta(Z) = 0, \quad \overline{\theta^1}(Z) = 0$$

which we make unique by adding the equation

$$\theta^1(Z) = 1.$$

The dual frame is given by $\{T, Z, \overline{Z}\}$ where T is given by (1.6) and

$$Z = \overline{\lambda}(Z_0 - \overline{\mu}\overline{Z_0}). \tag{3.4}$$

We extend the Euclidean inner product on $\mathbf{R}^4$ to a bi-linear form on the complexification by taking it to be conjugate linear in the second factor, $< X + iY, U + iV >=< X, U > + < Y, V > +i < Y, U > -i < X, V >$. We also normalize the metric so that

$$\|Z_0\|^2 =< Z_0, Z_0 >= 1.$$

Note that

$$< Z_0, \overline{Z_0} >= 0$$

and that

$$\begin{aligned} \|Z\|^2 &= |\lambda|^2(1+|\mu|^2) \\ < Z, \overline{Z} > &= -2\overline{\mu}\overline{\lambda}^2. \end{aligned}$$

We first restrict μ to satisfy $|\mu| < 1$. Note that if $|\mu| = 1$ then Z and $\overline{Z}$ are dependent and we do not have a CR structure. It is clear that we may take λ to be real and positive without changing the pseudo-hermitian structure. So we rewrite (3.3) as

$$\lambda^2(1-|\mu|^2) = 1.$$

To start the derivation of the structural equations, we solve (3.2) for θ_0^1,

$$\theta_0^1 = \lambda\theta^1 - \mu\lambda\overline{\theta^1}.$$

Thus

$$\begin{aligned} d\theta^1 &= \lambda(d\theta_0^1 + \mu d\overline{\theta_0^1}) \\ &= \lambda^2(1+|\mu|^2)\theta^1 \wedge \omega - 2\mu\lambda^2(\overline{\theta^1} \wedge \omega) \end{aligned}$$

Using the structure equations for the standard structure, we obtain

$$d\theta^1 = \theta^1 \wedge \left(-2i\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right)\right)\theta - \frac{4i\mu}{1-|\mu|^2}\theta \wedge \overline{\theta^1}.$$

We see that the torsion is

$$A = \frac{-4i\mu}{1-|\mu|^2} \tag{3.5}$$

and for the connection form

$$\omega = -2i\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right)\theta_0 \tag{3.6}$$

we have

$$d\omega = -2i\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right)i\theta^1 \wedge \overline{\theta^1} \tag{3.7}$$

and so the curvature is

$$R = 2\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right). \tag{3.8}$$

We note that R and $A\overline{\theta^1}$ are invariant in the following sense (and for general pseudo-hermitian structures, not only for left-invariant structures). Fix a pseudo-hermitian structure and compute these quantities with repect to some choice of θ^1. Now make another choice, call it $\tilde{\theta^1}$ to obtain $\tilde{R}$ and $\tilde{A}$. At first, we have only that $\tilde{\theta^1} = \alpha\theta^1 + \beta\theta$ with $\alpha \neq 0$. but from $d\theta = i\theta^1\overline{\theta^1} = i\tilde{\theta^1}\overline{\tilde{\theta^1}}$, we see that $|\alpha| = 1$ and $\beta = 0$. Next, we write the structure equations for θ and $\tilde{\theta^1}$. This gives us $\tilde{\omega}, \tilde{A}$ and $\tilde{R}$. We then substitute $\tilde{\theta^1} = \alpha\theta^1$ and compare with the structure equations for θ and θ^1. Thus

$$\begin{aligned} \omega &= \tilde{\omega} + \overline{\alpha}d\alpha \\ A &= \tilde{A}\overline{\alpha}^2 \\ R &= \tilde{R}. \end{aligned}$$

If $|\mu| > 1$ then the pseudo-hermitian structure is negatively oriented. So we consider $(H, J, -\theta_0)$ and compute the values for the torsion and the curvature. We still have $\theta^1 = \lambda(\theta_0^1 + \mu\overline{\theta_0^1})$ but now $\theta = \theta_0$ is replaced by $-\theta$. From

$$\theta^1 \wedge \overline{\theta^1} = \lambda^2(1 - |\mu|^2)\theta_0^1 \wedge \overline{\theta_0^1},$$

we have

$$d(-\theta) = i\theta^1 \wedge \overline{\theta^1}$$

provided

$$\lambda^2(1 - |\mu|^2) = -1.$$

That is

$$\lambda^2 = \frac{1}{|\mu| - 1}.$$

This leads to the equation

$$d\theta^1 = \theta^1 \wedge \omega + A\theta \wedge \theta^1$$

with

$$\omega = -2i\frac{|\mu|^2 + 1}{|\mu|^2 - 1}, \text{ and } A = \frac{4i\mu}{|\mu|^2 - 1}$$

and the equation

$$d\omega = R\theta^1 \wedge \overline{\theta^1} - 2i\Im V\overline{\theta^1} \wedge (-\theta)$$

with

$$R = \frac{|\mu|^2 + 1}{|\mu|^2 - 1}.$$

4. Geodesics

We compute the Hamiltonian flow in the co-tangent bundle for a general pseudo-hermitian structure and project down to find the geodesics on S^3. Then we will specialize to left invariant structures, especially $\mu = 0$. This latter case was studied using different techniques in [4].

As usual we use the frame $Z, \overline{Z}, T$ and the dual coframe $\theta, \theta^1, \overline{\theta^1}$. For coordinates on T^*M we use

$$(x; \zeta, \eta) \to (x; \zeta\theta^1 + \overline{\zeta\theta^1} + \eta\theta)$$

evaluated at x.

The real two-form

$$\begin{aligned} -\omega_S &= d(\zeta\theta^1 + \overline{\zeta\theta^1} + \eta\theta) \\ &= d\zeta \wedge \theta^1 + d\overline{\zeta} \wedge \overline{\theta^1} + d\eta \wedge \theta \\ &\quad +\zeta(\theta^1 \wedge \omega + A\theta \wedge \overline{\theta^1}) + \overline{\zeta}(\overline{\theta^1} \wedge (-\omega) + \overline{A}\theta \wedge \theta^1) + \eta i \theta^1 \wedge \overline{\theta^1} \end{aligned}$$

is a global symplectic form on T^*S^3. Following the approach of Strichartz [8], we take as a Hamiltonian

$$H = |\zeta|^2.$$

The associated Hamiltonian vector field X_H is defined by using interior multiplication

$$X_H \lrcorner \omega_S = dH.$$

A short computation yields

$$X_H = \overline{\zeta} Z + \zeta \overline{Z} + B\partial_\zeta + \overline{B}\partial_{\overline{\zeta}} + C\partial_\eta$$

with

$$\begin{aligned} B &= i\eta\zeta + \zeta\omega(X) \\ C &= \zeta^2 A + \overline{\zeta}^2 \overline{A}. \end{aligned}$$

This computation is based on the observations that

$$Z \lrcorner d(-\omega_S) = -d\zeta - \zeta\omega(Z)\theta^1 + \zeta\omega - \overline{\zeta A}\theta + i\eta\overline{\theta^1} + \overline{\zeta}\omega(Z)\overline{\theta^1}$$

and

$$\begin{aligned} \partial_\zeta \lrcorner (-\omega_S) &= \theta^1 \\ \partial_\eta \lrcorner (-\omega_S) &= \theta. \end{aligned}$$

Let $\Gamma(t)$ be any integral curve of X_H, $\gamma(t) = \pi(\Gamma(t))$ its projection down to S^3 and $X(t) = \pi_* X_H(t)$ the tangent to $\gamma(t)$.

Note that $X = \overline{\zeta} Z + \zeta \overline{Z}$. We claim that X satisfies the geodesic equations as in, for example, [1] and [7].

Theorem 4.1. *The curve $\gamma(t)$ is Legendrian and its tangent $X(t)$ satisfies*

$$\nabla_X X = aJX \text{ and } Xa = < Tor(X,T), X > .$$

Here $Tor(X,T)$ is defined as usual by

$$Tor(X,T) = \nabla_X T - \nabla_T X - [X,T].$$

Proof. Recall that a curve is said to be Legendrian if its tangent vector always lies in the contact distribution. This is clearly the case here.

We will need to compute terms like $X(\zeta)$ along the given curve $\gamma(t)$. For any function f on $\Gamma \subset T^*(S^3)$ we have

$$\Gamma^* f = \gamma^*(\pi^{-1})^* f$$

and so

$$(\Gamma_* \frac{d}{dt}) f = \gamma_* \frac{d}{dt}(f \circ \pi^{-1}).$$

Thus, in particular

$$X_H \zeta = X(\zeta \circ \pi^{-1}).$$

With only a slight abuse of notation, we identify ζ and $\zeta \circ \pi^{-1}$ and write

$$\begin{aligned}
\nabla_X X &= \nabla_X(\bar{\zeta} Z + \zeta \overline{Z}) \\
&= X(\bar{\zeta}) Z + \bar{\zeta} \nabla_X Z + X(\zeta) \overline{Z} + \zeta \nabla_X \overline{Z} \\
&= X_H(\bar{\zeta}) Z + X_H(\zeta) \overline{Z} + \bar{\zeta} \omega(X) Z - \zeta \omega(X) \overline{Z} \\
&= \overline{B} Z + B \overline{Z} + \omega(X)(\bar{\zeta} Z - \zeta \overline{Z}) \\
&= (-\bar{\zeta} \omega(X) - i\eta \bar{\zeta}) Z + (\zeta \omega(X) + i\eta \zeta) \overline{Z} + \omega(X)(\bar{\zeta} Z - \zeta \overline{Z}) \\
&= \eta(-i\bar{\zeta} Z + i\zeta \overline{Z})
\end{aligned}$$

Note that

$$JX = J(\bar{\zeta} Z + \zeta \overline{Z}) = i(\bar{\zeta} Z - \zeta \overline{Z})$$

and so we have

$$\nabla_X X = -\eta JX.$$

Recall that the connection is defined by

$$\begin{aligned}
\nabla Z &= \omega \otimes Z \\
\nabla \overline{Z} &= -\omega \otimes \overline{Z} \\
\nabla T &= 0.
\end{aligned}$$

So

$$\begin{aligned}
Tor(X,T) &= \nabla_X T - \nabla_T X - [X,T] \\
&= -\nabla_T X - [X,T] \\
&= -\nabla_T(\bar{\zeta} Z + \zeta \overline{Z}) - [X,T] \\
&= -\bar{\zeta} \omega(T) Z + \zeta \omega(T) \overline{Z} - [X,T].
\end{aligned}$$

Also

$$\begin{aligned}[X,T] &= [\overline{\zeta}Z + \zeta\overline{Z}, T] \\ &= \overline{\zeta}(-\omega(T)Z + \overline{AZ}) + \zeta(-\overline{\omega}(T)\overline{Z} + AZ).\end{aligned}$$

So

$$\begin{aligned}Tor(X,T) &= -\{\overline{\zeta}\omega(T)Z - \zeta\omega(T)\overline{Z}\} - \{\overline{\zeta}(-\omega(T)Z + \overline{AZ}) + \zeta(\omega(T)\overline{Z} + AZ)\} \\ &= -\{\zeta AZ + \overline{\zeta AZ}\}.\end{aligned}$$

Finally,

$$\begin{aligned}< Tor(X,T), X > &= < -\{\zeta AZ + \overline{\zeta AZ}, X > \\ &= -\{A\zeta^2 + \overline{A\zeta}^2\} \\ &= -C\end{aligned}$$

and

$$X(-\eta) = X_H(-\eta) = -C =< Tor(X,T), X > .$$

□

The projection of an integral curve of X_H, that is, a curve satisfying the equations of Theorem 4.1, is called a geodesic and the equations themselves are called the geodesic equations. We now look more closely at them. We have

$$\gamma'(t) = \overline{\zeta(t)}Z(\gamma(t)) + \zeta(t)\overline{Z(\gamma(t))} \tag{4.1}$$

$$\zeta'(t) = i\eta\zeta + \omega(\gamma')\zeta \tag{4.2}$$

$$\eta'(t) = 2\mathfrak{R}(\zeta^2 A). \tag{4.3}$$

We start our analysis of these equations with two simple observations. First, since $\omega(\gamma')$ is imaginary the second equation implies that $|\zeta|$ is a constant along each orbit. Second, the two curves $\Gamma(t)$ through the points $(\gamma_0, \zeta_0, \eta_0)$ and $(\gamma_0, a\zeta_0, a\eta_0)$ with a real and nonzero project to the same curve $\gamma(t)$.

To study the geodesics on S^3 we make use of the coordinates on $\mathbf{C}^2$. So we write

$$\gamma(t) = (\alpha(t), \beta(t))$$

with $|\alpha|^2 + |\beta|^2 = 1$. Then

$$\gamma'(t) = 2\mathfrak{R}(\alpha'(t)\partial_z + \beta'(t)\partial_w)$$

In order to derive useful equations for α and β we need now to restrict to the left invariant structures. Using (1.3) and (3.4) this leads to

$$\alpha' = \sigma\overline{\beta} \tag{4.4}$$

$$\beta' = -\sigma\overline{\alpha} \tag{4.5}$$

with $\sigma = \overline{\lambda\zeta} - \zeta\lambda\mu$.

Note that $|\alpha|^2 + |\beta|^2$ is indeed constant.

We have

$$\begin{aligned} \|\gamma'\|^2 &= \left\|\overline{\zeta}Z + \zeta\overline{Z}\right\|^2 \\ &= |\zeta|^2 < Z, Z > +|\zeta|^2 < \overline{Z,Z} > \\ &\quad +\overline{\zeta}^2 < Z, \overline{Z} > +\zeta^2 < \overline{Z}, Z > \\ &= \lambda^2(2|\zeta|^2(1+|\mu|^2) - 2\overline{\zeta}^2\overline{\mu} - 2\zeta^2\mu). \end{aligned}$$

Note that the length of the velocity vector for a geodesic is always a constant only if $\mu = 0$. That is, it is always constant only for the standard structure on the sphere.

We now easily derive a second order differential equation for α:

$$\alpha'' = \frac{\sigma'}{\sigma}\alpha' - |\sigma|^2\alpha.$$

We limit ourselves to the pseudo-hermitian structures with $|\mu| < 1$. Recall that we may take λ to be positive and that

$$\lambda^2 = \frac{1}{1 - |\mu^2|}.$$

Also, we have

$$A = \frac{-4i\mu}{1-|\mu^2|} = -4i\mu\lambda^2 \text{ and } \omega(X) = 0.$$

We use the next lemma to relate η to the curvature. Let $\phi(t) = x(t) + iy(t)$ be any smooth curve.

Lemma 4.1. *The curvature of the plane curve (x(t),y(t)) is*

$$\kappa(t) = \frac{|\Im(\overline{\phi'}(t)\phi'')|}{|\phi'(t)|^3}. \tag{4.6}$$

Proof. The classic expression of the curvature is

$$\kappa = \frac{|x'y'' - y'x''|}{((x')^2 + (y')^2)^{\frac{3}{2}}}.$$

Now just use $\Im(\xi\eta) = \Re\xi\Im\eta + \Im\xi\Re\eta$. □

As an illustration of the lemma, we see that if the curve is parametrized by arc length (which is not the case in our application) then $\overline{\phi'}\phi''$ is imaginary and so, as expected, $\kappa(t) = |\phi''(t)|$.

Lemma 4.2. *Let $\gamma(t)$ be a geodesic for a left invariant structure. Let $\alpha(t)$ be the projection of $\gamma(t)$ into the complex tangent line at P. Then the curvature of $\alpha(t)$ at $t = t_0$ is equal to*

$$\frac{|\eta(t_0)|\ |\zeta(t_0)|^2}{|\lambda\overline{\zeta(t_0)} - \lambda\mu\zeta(t_0)|^3}.$$

Proof. We may assume that $P = (0,1)$. Thus $\alpha(t_0) = 0$ and $\beta(t_0) = 1$. Thus from (4.4) and (4.5) we have, at t_0

$$\alpha' = \sigma$$

and

$$\alpha'' = \sigma'.$$

It follows that

$$\Im(\overline{\alpha}'\alpha'') = -\eta|\zeta|^2.$$

(Recall that $\lambda^2(1 - |\mu|^2) = 1$ and $\omega(X) = 0$ for $X \in H$.) Our result follows. □

Note that for a fixed $\zeta(t_0)$, η measures the curvature.

It is curious that now the equations for ζ' and η' are independent of the base point γ, since the term $\omega(\gamma')$ is 0. We have a singular foliation of $C \times R$ which, in some sense, completely determines the geodesics on S^3 and depends only on the torsion A.

Finally we consider the standard structure and study the geodesics in some detail. Now we have $\mu = 0$ and $\lambda = 1$, in addition to $\omega(\gamma') = 0$, and since the torsion is now zero, (4.3) becomes

$$\eta = \text{ constant}$$

So (4.2) may be solved

$$\zeta = ae^{i(\eta t + \phi)}$$

where ϕ and a are also constants and $a > 0$. Note that, in addition to measuring the curvature of $\alpha(t)$, η may be interpreted as the speed at which ζ rotates along the circle $|\zeta| = a$. In place of (4.4) and (4.5) we have

$$\alpha' = \overline{\zeta\beta} \tag{4.7}$$

$$\beta' = -\overline{\zeta\alpha}. \tag{4.8}$$

We start our discussion of curvature in the standard structure by relating η to the curvature of $\gamma(t)$ rather than to its projection $\alpha(t)$.

Lemma 4.3. *Each geodesic for the standard structure has constant curvature given by*

$$\kappa = \frac{\sqrt{1+\eta^2}}{2}.$$

Proof. We start with

$$\begin{aligned} \alpha' &= \overline{\zeta\beta} \\ \beta' &= -\zeta\overline{\alpha} \\ \zeta' &= i\eta\zeta \end{aligned}$$

with $|\zeta| = 1$ and η a constant. Since $|\gamma'| = 1/\sqrt{2}$ the usual formula for curvature of a space curve

$$\kappa = \frac{1}{|\gamma'|}\left|\frac{d}{dt}\frac{\gamma'}{|\gamma'|}\right|$$

gives us

$$\kappa = \frac{1}{2}|\gamma''|$$

Since we have

$$\begin{aligned} \alpha'' &= -i\eta\overline{\zeta\beta} - \alpha \\ \beta'' &= i\eta\overline{\zeta}\overline{\alpha} - \beta, \end{aligned}$$

this leads, after a short calculation, to the desired equality. □

We may simplify further

$$\begin{aligned} \alpha'' &= -i\eta\overline{\zeta\beta} + \overline{\zeta}(-\zeta\alpha) \\ &= -i\eta\alpha' - \alpha. \end{aligned}$$

The same equation is satisfied by β:

$$\beta'' = -i\eta\beta' - \beta.$$

The general solution to

$$\alpha'' + i\eta\alpha' + \alpha = 0$$

is

$$\alpha = e^{-i\eta t/2}(C_1 e^{i\nu t} + C_2 e^{-i\nu t})$$

with

$$\nu = \frac{\sqrt{\eta^2 + 4}}{2}. \tag{4.9}$$

We now consider only geodesics starting at the point $(0,1) \in S^3 \subset \mathbf{C}^2$ that are the projections of the orbits of X_H with $|\zeta| = 1$.That is, we want the solution to (4.7) and (4.8), where $\zeta(t)$ is given by (4.2), satisfying the initial conditions

$$\alpha(0) = 0 \text{ and } \beta(0) = 1.$$

Since $\zeta(0) = e^{i\phi}$,

$$\alpha'(0) = e^{-i\phi}$$

and

$$\beta'(0) = 0.$$

We record this as a lemma.

Lemma 4.4. *If* $\{\alpha(t), \beta(t)\}$ *solves*

$$\begin{aligned} \alpha' &= \overline{\zeta\beta} \\ \beta' &= -\overline{\zeta\alpha} \end{aligned}$$

$$\alpha(0) = 0 \text{ and } \beta(0) = 1$$

with $\zeta = e^{i(\eta t+\phi)}$ *and* η *and* ϕ *constant, then* $\alpha(t)$ *solves*

$$\alpha'' + i\eta\alpha' + \alpha = 0.$$

with initial conditions

$$\alpha(0) = 0 \text{ and } \alpha'(0) = e^{-i\phi}.$$

In addition

$$\alpha''(0) = -i\eta e^{-i\phi}.$$

Conversely, if $\alpha(t)$ *solves*

$$\alpha'' + i\eta\alpha' + \alpha = 0$$

$$\alpha(0) = 0, \quad \alpha'(0) = e^{-i\phi}$$

and we set $\zeta = e^{i(\eta t+\phi)}$ *and* $\beta(t) = \overline{\zeta\alpha'}$*, then* $\{\alpha(t), \beta(t)\}$*solves*

$$\alpha' = \overline{\zeta\beta} \tag{4.10}$$

$$\beta' = -\overline{\zeta\alpha} \tag{4.11}$$

with

$$\alpha(0) = 0 \text{ and } \beta(0) = 1.$$

Further, $\alpha'(0) = e^{-i\phi}$*,* $\beta'(0) = 0$*, and in addition* $\beta''(0) = -1$*.*

So for any geodesic starting at $(0,1)$ we have

$$\begin{aligned} \gamma(0) &= (0,1) \\ \gamma'(0) &= (e^{-i\phi}, 0) \\ \gamma''(0) &= (-i\eta e^{-i\phi}, -1) \end{aligned} \tag{4.12}$$

and these initial conditions determine the geodesic uniquely. That is, we prescribe the initial point and direction and also the curvature as measured by η.

The circle acts on the space of geodesics by mapping $\gamma(t) = (\alpha(t), \beta(t))$ to $\gamma_1(t) = (e^{i\psi}\alpha(t), \beta(t))$. It is easily verified that γ_1 is indeed a geodesic. Specifically, it is the geodesic obtained by replacing ϕ by $\phi + \psi$ in the initial conditions.

Continuing the consideration of geodesics starting at the point $(0,1)$ we note that the solution to

$$\alpha'' + i\eta\alpha' + \alpha = 0$$

with the initial conditions

$$\alpha(0) = 0 \text{ and } \alpha'(0) = e^{-i\phi}$$

is

$$\alpha(t) = \frac{1}{\nu} e^{-i(\eta t/2+\phi)} \sin(\nu t) \tag{4.13}$$

with ν given by (4.9). We compute the corresponding β

$$\beta = \overline{\zeta\alpha'}$$

with

$$\zeta = e^{-i(\eta t+\phi)}$$

to obtain

$$\beta(t) = \frac{i\eta}{2} e^{-i(\eta t+\phi)} \overline{\alpha(t)} + e^{-i\eta t/2} \cos(\nu t). \tag{4.14}$$

Let $\mathcal{C}$ denote the unit circle in the w plane

$$\mathcal{C} = \{(0,w) : \ |w| = 1\}.$$

Fix η and ϕ. The corresponding geodesic intersects $\mathcal{C}$ in the sequence

$$w_N = (-1)^N e^{-iN\theta}$$

for $N = 0, 1, \ldots$ and $\theta = \eta\pi/2\nu$. This can be seen directly from (4.13) and (4.14) or by using the $SU(2)$ invariance. From this is follows that the periodicity of the geodesic depends on η but not on ϕ (as is clear also from the circle action on the space of geodesics). In particular the geodesic starting at $(0,1)$ and with initial conditions (4.12) is periodic if and only if η/ν is rational. That is, periodicity holds if and only if

$$\frac{\eta}{\sqrt{\eta^2+4}} \tag{4.15}$$

is rational.

In somewhat more detail, assume $\eta/\nu = p/q$ and set $T = 4\pi q/\nu$. Then $\alpha(T) = 0$ and $\beta(T) = 1$. Since we have $\zeta(t) = e^{i(\eta t+\phi)}$ and (4.10) and (4.11), it follows that

$$\alpha'(T) = e^{-i\phi} \text{ and } \beta'(T) = 0.$$

Fix some θ, $-\pi < \theta < \pi$, and set $w_0 = -e^{i\theta}$. We want to determine all the geodesics which start at $(0,1)$ and go through $(0,w_0) \in \mathcal{C}$. We see that ϕ is arbitrary and η is given by

$$\eta = \frac{-2\theta}{\sqrt{\pi^2 - \theta^2}}. \tag{4.16}$$

Thus there are infinitely many distinct geodesics connecting $(0,1)$ and $(0,w_0)$. These geodesics are periodic precisely when θ/π is rational. This follows immediately from (4.15).

Since we have parametrized our geodesics by arc length we see that all these geodesics have length

$$t_0 = \sqrt{\pi^2 - \theta^2}. \tag{4.17}$$

We call w_0 the point antipodal to $(0,1)$ along this 2-sphere. Note that $\theta = \pi$ corresponds to $w_0 = 1$. Also note that the real line $-\infty < \eta < \infty$ is mapped to the circle with this point deleted, $-\pi < \theta < \pi$. For $\eta = 0$, the geodesics are the usual geodesics for $S^2 \subset S^3 \subset R^4$ where

$$S^2 = S^3 \cap \{x_4 = 0\}. \tag{4.18}$$

So all the geodesics starting at $(0,1)$ and having $(0,w_0)$ as their first intersection with $\mathcal{C}$ have length given by (4.17) and form a 2-sphere. Now follow these geodesics but consider w_0 as the starting point. The next intersection is when

$$w_1 = -e^{i\theta} w_0 = e^{2i\theta}.$$

So for each value of η , equivalently each value of θ, we have a chain of 2-spheres. Since every geodesic starting at $(0,1)$ intersects $\mathcal{C}$ again, and since S^3 is geodesicly connected, as η varies, these 2-spheres fill out all of S^3. Can these 2-spheres intersect at any other points besides these "antipodal" points? If so, it would be interesting to determine the curve along which they intersect.

References

[1] E. Barletta, S. Dragomir, *Jacobi fields of the Tanaka-Webster connection on Sasakian manifolds*, Kodai Math. J. 29 (2006), no. 3, 406-454.

[2] É. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes I*, Ann. Math. Pura Appl. **11.4** (1932), 17–90.

[3] É. Cartan, *Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes II*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 2e série **1.4** (1932), 333–354.

[4] D-C. Chang, I. Markina, A. Vasil'ev, *Sub-Riemannian geodesics on the 3-D sphere*, Complex Anal. Oper. Theory 3 (2009), no. 2, 361-377.

[5] H. Poincaré, *Les fonctions analytiques de deux variables et la représentation conforme*, Rendiconti del Circolo Matematico di Palermo **23.1** (1884-1940), 185–220.

[6] S.S. Chern, J. Moser, *Real hypersurfaces in complex manifolds*, Acta mathematica **133.1** (1974), 219–271

[7] M. Rumin, *Formes diffrentielles sur les varits de contact*, J. Differential Geom. 39 (1994), no. 2, 281-330.

[8] R. Strichartz, *Sub-Riemannian Geometry*, J. Differential Geom. 24 (1986),221-263.

[9] N. Tanaka, A differential geometric study on strongly pseudo-convex manifolds, Lectures in Mathematics, Department of Mathematics, Kyoto University, No. 9. Kinokuniya Book-Store Co. Ltd., Tokyo, 1975.

[10] S. M. Webster, *Pseudo-Hermitian structures on a real hypersurface*, J. Differential Geom. 13 (1978), no. 1, p. 25-41.