Left-invariant CR structures on 3-dimensional Lie groups

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A *CR structure* on a 3-dimensional manifold M is a rank 2 subbundle $D \subset TM$ together with an almost complex structure J on D

$$J : D \to D, \quad J^2 = -Id \ .$$

Equivalently, a CR structure is a complex line subbundle $V \subset TM \otimes \mathbf{C}$ such that $V \cap \overline{V} = \{0\}$.

$$D = \{\Re V, \Im V\}$$

Notation: (M, V) or (M, L).
Usually assume $L,\ \bar{L},\ [L, \bar{L}]$ linearly independent (spc).
Note: $[L, \bar{L}] \equiv 0$ means that M is foliated by complex curves.
$f : M \to \mathbf{C}$ is a CR function if $Lf = 0$ for each section of V.

G a 3 dimensional Lie group
$\mathfrak{G}$ its Lie algebra

$$\mathfrak{G} = T_e G = \{L : g_* L_e = L_g\}$$

(G, L) is a CR manifold provided

$$L \cap \bar{L} = \{0\}.$$

Conjugation

$$\Phi : G \to Aut(G)$$

$$\Phi(g)h = ghg^{-1}$$

induces a map

$$Ad : G \to Aut(\mathfrak{G})$$

with derivative at the identity

$$ad : \mathfrak{G} \to \mathfrak{G}$$

given by

$$ad(X)Y = [X, Y].$$

Immersions into $\mathbf{CP}^2$

A complex line sub-bundle $V \subset T(G) \otimes \mathbf{C}$ determines a point in $P(\mathfrak{G} \otimes \mathbf{C}) = \mathbf{CP}^2$, $L \to [L] \in P(\mathfrak{G} \otimes \mathbf{C})$, and the orbit of L

$$\{X : \exists g \in G, \;\; Ad_g L = X\}$$

gives a map of G into $\mathbf{CP}^2$

$$\phi : G \to \mathbf{CP}^2$$

$$g \to [Ad_g L]$$

with differential at the identity

$$\phi_* X = [[X, L]].$$

ϕ is always a CR map

$$G \to \mathbf{CP}^2.$$

ϕ is an immersion if

$$[X, L] = 0 \text{ for } X \text{ real } \Rightarrow X = 0.$$

Spherical and Aspherical

(M, V) is spherical near a point if it is locally CR equivalent to the standard CR structure on S^3.

Theorem (Cartan 1932)

The local CR equivalence group of a 3 dimensional aspherical CR manifold has dimension at most 3.

Consider two left-invariant aspherical CR structures (G_i, V_i) on two connected 3-dimensional Lie groups G_i

Corollary

If the two CR structures are locally equivalent, then there exists a group isomorphism $G_1 \to G_2$ which extends this local CR equivalence.

Proof.

1. Right-invariant vector fields generate left-translations. So $\mathcal{R}_{G_i} \subset Aut_{CR}(G_i, V_i)$.
2. Since the dimension are equal these groups coincide near the identity.
3. The local CR equivalence f maps $\mathcal{R}_{G_1} \to \mathcal{R}_{G_2}$.
4. f preserves the group structure.

□

The most interesting examples: $SL(2,\mathcal{R})$ and $SU(2)$

Basis for $sl(2,\mathcal{R})$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Basis for $su(2)$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Note

$$sl(2,\mathbf{C}) = su(2) \otimes \mathbf{C}.$$

and

$$P(\mathfrak{G} \otimes \mathbf{C}) = P(sl(2,\mathbf{C})) = \mathbf{CP}^2.$$

Geometry of $P(sl(2,\mathbf{C}))$

Let $L \in sl(2,\mathbf{C})$ be given as

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

The Killing form is

$$K(L,L) = a^2 + bc = -detL$$

and

$$K(L,L) = 0$$

defines a conic in $\mathcal{C} = \mathbf{CP}^1 \subset P(sl(2,\mathbf{C}))$. The polar line with respect to $\mathcal{C}$ of a point $[L] \in \mathbf{CP}^2$ determines two points in $\mathcal{C}$.

$$\mathcal{S} = \{\{\zeta_1, \zeta_2\} : \zeta_i \in \mathbf{CP}^1\}.$$

$$L \to [L] \in P(sl(2, \mathbf{C})) \to \mathcal{S}$$

is $SL(2, \mathbf{C})$ equivariant.

$$\begin{array}{ccc} P(sl(2,\mathbf{C})) & \xrightarrow{Ad_g} & P(sl(2,\mathbf{C}))) \\ \downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow[g]{} & \mathcal{S} \end{array}$$

Classifying the orbits of $[L]$ under $SL(2, \mathcal{R})$ or $SU(2)$ uses that

- $SL(2, \mathcal{R})$ preserves the hyperbolic distance in the upper half-plane.
- $SU(2)$ preserves the Euclidean distance in S^2.

So the following are equivalent:

1. $[L]$ and $[\tilde{L}]$ are in the same Aut_G orbit.
2. $d(\zeta_1, \zeta_2) = d(\tilde{\zeta}_1, \tilde{\zeta}_2)$.
3. The left-invariant CR structures (G, L) and $(G, \tilde{L})$ are CR equivalent under a group automorphism.

Theorem

Let $V_t \subset T_{\mathbf{C}}SL(2,\mathcal{R})$, $t \in (-1,1]$ *be the left-invariant complex line bundle spanned at e by*

$$L_t = \begin{pmatrix} i\frac{1+t}{2} & t \\ 1 & -i\frac{1+t}{2} \end{pmatrix} \in sl_2(\mathbf{C}).$$

Then

1. V_t is a left-invariant CR structure for all $t \neq 0$.
2. V_0 is a foliation by complex curves.
3. V_t is spherical if $t = 1$ or $-3 \pm 2\sqrt{2}$ and aspherical otherwise.
4. Every left-invariant CR structure on $SL_2(\mathcal{R})$ is CR equivalent to V_t for a unique t.
5. The aspherical left-invariant CR structures V_t, $t \in (-1,1) \setminus \{0, -3+2\sqrt{2}\}$, are pairwise non-equivalent, even locally.

The immersion results rely on a geometric observation. Recall the null cone of the Killing form of $sl(2,\mathcal{R})$

$$\mathcal{C} = \{L : a^2 + bc = 0\}.$$

A left-invariant CR structure on $SL(2,\mathcal{R})$ is called hyperbolic if the real 2-plane spanned by $\{\Re L, \Im L\}$ intersects $\mathcal{C}$ in two lines. It is called elliptic if the intersection is the origin.
Note: One spherical structure is elliptic, the other is hyperbolic.

Theorem

1. *The elliptic left-invariant spherical CR structure on* $SL(2,\mathcal{R})$ *(*$t=1$*) is realizable by the map*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} b+ia \\ d+ic \end{pmatrix}.$$

   It is also realizable as the complement of $S^3 \cap \mathbf{C}^1$*.*

2. *The hyperbolic spherical structure on* $SL(2,\mathcal{R})$ *(*$t=-3+2\sqrt{2}$*) is realizable as the complement of* $S^3 \cap \mathcal{R}^2$*.*
3. *The rest of the left-invariant CR structures on* $SL(2,\mathcal{R})$ *are either* $4:1$ *covers, in the aspherical elliptic case* $0<t<1$*, or* $2:1$ *covers, in the aspherical hyperbolic case* $-1<t<0$*, of the orbits of* $SL(2,\mathcal{R})$ *in* $\mathbf{P}(sl(2,\mathbf{C}))$*.*

Theorem

Let $V_t \subset T_{\mathbf{C}}SU(2)$, $t \geq 1$ be the left-invariant complex line bundle spanned at $e \in SU(2)$ by

$$L_t = \begin{pmatrix} 0 & t-1 \\ t+1 & 0 \end{pmatrix}.$$

1. *V_t is a left-invariant CR structure on $SU(2)$ for all $t \geq 1$.*
2. *V_t is spherical if and only if $t = 1$.*
3. *Every left-invariant CR structure on $SU(2)$ is CR equivalent to V_t for a unique t.*
4. *The aspherical left-invariant CR structures V_t, $t > 1$, are pairwise non-equivalent, even locally.*
5. *V_1 is realized by the usual CR structure on $S^3 \subset \mathbf{C}^2$. The aspherical structures are realized as $4:1$ covers of the adjoint orbits of $SU(2)$ in $\mathbf{P}(\mathfrak{sl}(2, \mathbf{C}))$.*

The Heisenberg Group

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathcal{R} \right\}$$

and

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathcal{R} \right\}$$

H is isomorphic to the group on the hyperquadric

$$\{(z, w) \in \mathbf{C}^2 : \Im w = |z|^2\}.$$

Theorem

1. *Every plane in* $\mathfrak{h}$ *containing*

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   gives a foliation of H *by complex curves.*

2. *Every other plane is in the orbit of*

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}$$

   under $Aut(H)$*.*

3. L *gives a spherical CR structure equivalent to the usual one on the hyperquadric.*

Observation

Let $V_t, t \in \mathcal{R}$ be a smooth family of CR structures on some M^3, such that

$$V_t \equiv V_s \text{ for } s \neq 0,\ \ t \neq 0$$

but

$$V_1 \not\equiv V_0.$$

Then V_0 is a foliation of M by complex curves.

Example

$$L_t = \begin{pmatrix} 0 & i & 1 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} o & i & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\{L_t, \bar{L_t}\}$ is not involutive for $t \neq 0$ but $\{L_0, \bar{L_0}\}$ is involutive.

The Euclidean Group

$$E = \left\{ \begin{pmatrix} \cos\theta & \sin\theta & u \\ -\sin\theta & \cos\theta & v \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$\mathfrak{e} = \left\{ \begin{pmatrix} 0 & x & y \\ -x & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

Theorem

Let $V \subset T_{\mathbf{C}}E$ be the left-invariant line bundle whose value at $e \in E$ is spanned by

$$L = \begin{pmatrix} 0 & -i & 1 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

1. *Every spc left-invariant CR structure on E is CR equivalent to V by $Aut(E)$.*
2. *V is a aspherical.*
3. *V is realized in $\mathbf{P}(\mathfrak{e} \otimes \mathbf{C}) = \mathbf{C}P^2$ by the adjoint orbit of $[L]$. This is CR equivalent to the real hypersurface $[\Re(z_1)]^2 + [\Re(z_2)]^2 = 1$ in $\mathbf{C}^2$.*

$$L_t = \begin{pmatrix} 0 & -it & a \\ it & 0 & bi \\ 0 & 0 & 0 \end{pmatrix}$$

is an aspherical CR structure for $t \neq 0$ but a foliation by complex curves for $t = 0$.

Cartan's Moving Frames

$$\begin{array}{ccc} H & \longrightarrow & SU(2,1) \\ & & \Big\downarrow \\ & & Q \end{array}$$

$\dim SU(2,1) = 8, \quad \dim H = 5.$
Maurer-Cartan form $\Theta : TSU(2,1) \to \mathfrak{su}(2,1)$

Let M be a three dimensional (spc) CR structure.

$$\begin{array}{ccc} H^5 & \longrightarrow & B^8 \\ & & \Big\downarrow \\ & & M^3 \end{array}$$

$\Theta : TB \otimes \mathbf{C} \to \mathfrak{su}(2,1) \otimes \mathbf{C}.$

Theorem

With each spc CR 3-manifold M there is canonically associated a bundle $B \to M$ with Cartan connection $\Theta : TB \to \mathfrak{su}(2,1)$, satisfying

1. *The eight components of Θ are pointwise linearly independent.*
2. *(The CR structure equations) There exist functions $R, S : B \to \mathbf{C}$ such that*

$$\begin{aligned} d\theta &= i\theta_1 \wedge \bar{\theta}_1 - \theta \wedge (\theta_2 + \bar{\theta}_2), \\ d\theta_1 &= -\theta_1 \wedge \theta_2 - \theta \wedge \theta_3, \\ d\theta_2 &= 2i\, \theta_1 \wedge \bar{\theta}_3 + i\, \bar{\theta}_1 \wedge \theta_3 - \theta \wedge \theta_4, \\ d\theta_3 &= -\theta_1 \wedge \theta_4 - \bar{\theta}_2 \wedge \theta_3 - R\, \theta \wedge \bar{\theta}_1, \\ d\theta_4 &= i\, \theta_3 \wedge \bar{\theta}_3 - (\theta_2 + \bar{\theta}_2)\theta_4 + (S\, \theta_1 + \bar{S}\, \bar{\theta}_1) \wedge \theta. \end{aligned}$$

3. *(Spherical structures) M is spherical if and only if $R \equiv 0$.*
4. *Any local CR diffeomorphism of CR manifolds $f : M \to M'$ lifts uniquely to an map $\tilde{f} : B \to B'$ with $\tilde{f}^*\Theta' = \Theta$.*

$\phi, \ \phi_1$ is an adapted coframe for the CR structure V on M if

1. $$\phi \text{ is real.}$$
2. $$\phi(V) = 0, \quad \phi_1(V) = 0.$$
3. $$\phi \wedge \phi_1 \wedge \bar{\phi_1} \neq 0.$$
4. $$d\phi = i\, \phi_1 \wedge \bar{\phi_1}.$$

Each adapted coframe has a unique lift to $\sigma : M \to B$ such that $\sigma^* \theta = \phi$ and $\sigma^* \theta_1 = \phi_1$.

Pulling back the other components of Θ we obtain

$$
\begin{aligned}
d\phi &= i\phi_1 \wedge \bar{\phi}_1 - \phi \wedge (\phi_2 + \bar{\phi}_2),\\
d\phi_1 &= -\phi_1 \wedge \phi_2 - \phi \wedge \phi_3,\\
d\phi_2 &= 2i\,\phi_1 \wedge \bar{\phi}_3 + i\,\bar{\phi}_1 \wedge \phi_3 - \phi \wedge \phi_4,\\
d\phi_3 &= -\phi_1 \wedge \phi_4 - \bar{\phi}_2 \wedge \phi_3 - r\,\phi \wedge \bar{\phi}_1,\\
d\phi_4 &= i\,\phi_3 \wedge \bar{\phi}_3 + (s\,\phi_1 + \bar{s}\,\bar{\phi}_1) \wedge \phi.
\end{aligned}
$$

M is spherical if and only if the Cartan relative curvature invariant r is identically 0.

Lemma

Let M be a manifold with a CR structure given by a coframe ϕ, ϕ_1 *satisfying*

$$\begin{aligned} d\phi &= i\phi_1 \wedge \bar{\phi}_1, \\ d\phi_1 &= a\,\phi_1 \wedge \bar{\phi}_1 + b\,\phi \wedge \phi_1 + c\,\phi \wedge \bar{\phi}_1, \end{aligned} \tag{1}$$

for some complex constants a, b, c. *Then*

$$r = ic(\frac{|a|^2}{3} + \frac{3ib}{2}).$$

If a left-invariant CR structure has a coframe satisfying (**??**) then $a,\ b,\ c$ are constants and so the structure is spherical if and only if

$$c(\frac{|a|^2}{3} + \frac{3ib}{2}) = 0.$$