LOCALLY CR SPHERICAL THREE MANIFOLDS

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Abstract. Every open and orientable three manifold has a CR structure which is locally equivalent to the standard CR structure on S^3.

The standard CR structure on S^3 is the one induced by the usual complex structure on $\mathbf{R^4}$. It is defined by the complex tangent vector field

$$L \in \mathbf{C} \otimes TS^3 \subset T^{(0,1)}(\mathbf{C^2})$$

given by

$$L = w\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial \bar{w}}$$

which annihilates the restrictions of holomorphic functions. Let V_0 denote the complex line bundle generated by L.

For any 3-manifold M a CR structure is a pair (M, V) where V is a complex line sub-bundle of $\mathbf{C} \otimes TM$ satisfying $V \cap \bar{V} = \{0\}$. This structure is said to be locally spherical if for each point there exist an open neighborhood $\mathcal{O}_1$ and a CR diffeomorphism F of $\mathcal{O}_1$ onto an open subset $\mathcal{O}_0 \subset S^3$ such that $F_*(V) = V_0$. There are many results about compact locally spherical three manifolds (for instance, [1], [2], [3], [5], [6]) which show that these manifolds are special. The situation is quite different for open manifolds. Although there are some interesting constructions on special open manifolds, particularly in connection with complex hyperbolic geometry [4], [8], this easy observation seems not to be in the literature:

Theorem. *Every open and orientable three dimensional manifold admits a locally spherical CR structure.*

Remark 1. *The CR structure may be taken to be C^ω.*

We start with the standard CR structure on S^3, remove one point, and choose any diffeomorphism of this new manifold to $\mathbf{R^3}$. We end up with a locally spherical CR structure on $\mathbf{R^3}$. Any map

$$F : M \to \mathbf{R^3}$$

which is a local diffeomorphism may be used to pull this CR structure on $\mathbf{R^3}$ back to M. The result is a locally spherical CR structure on M. The existence of some F is a well-known topological fact, first proved by Whitehead [9]. See also Phillips [7], Corollary 8.2 and the references cited there.

Remark 2. *The conclusion of the theorem also holds for any open parallelizable manifold of odd dimension. In the same manner we can pull back the standard Riemannian metric on S^n to obtain a metric of constant positive curvature on any*

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open parallelizable manifold. This does not contradict Myers' Theorem since this metric is never complete.

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