Left-Invariant CR and Pseudo Hermitian Structures on $S^3$

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January 4, 2016
A CR structure on $M^3$ is a two-plane distribution $H \subset TM$ and a complex structure on each fiber.

$$J : H \to H \text{ with } J^2 = -I.$$  

We denote this structure by $(M, H, J)$. It is often useful to extend $J$ by complex linearity to a map

$$J : \mathbb{C} \otimes H \to \mathbb{C} \otimes H.$$  

Then $J$ is completely determined by the eigenspace corresponding to the eigenvalue $i$ (or to the eigenvalue $-i$). So a CR structure is just as well given by a complex line bundle $B \subset \mathbb{C} \otimes H$, 

$$B \cap \overline{B} = \{0\}.$$
It will be useful to work with the dual formulation. Choose some $\theta$ such that $\theta^\perp = H$. Choose some $\theta^1$ such that

$$X \in H \implies \theta^1(X + iJX) = 0 \text{ and } \theta^1(X - iJX) \neq 0.$$ 

$(\theta, \theta^1)$ is called a CR coframe. The CR structure is **strictly pseudoconvex** if $\theta \wedge d\theta \neq 0$. So $\theta$ is a contact form and $H$ is a contact distribution. A normalized coframe $(\theta, \theta^1)$ satisfies

$$d\theta = i\theta^1 \wedge \theta^1.$$
\((H, J)\) uniquely defines the CR structure. are not unique.

\[
\tilde{\theta} = r\theta \\
\tilde{\theta}^1 = \alpha \theta^1
\]

with constants \(r\) real and \(\alpha\) complex, \(|\alpha|^2 = r > 0\) is also a normalized coframe.

A **pseudo-hermitian structure** is a strictly pseudoconvex \((H, J)\) and a choice of \(\theta\). If \((\theta, \theta^1)\) is a normalized coframe, then the only other normalized choices are

\[
\tilde{\theta} = \theta \\
\tilde{\theta}^1 = \lambda \theta^1
\]

with \(|\lambda| = 1\).
Given two CR structures \((M, H, J)\) and \((M, \tilde{H}, \tilde{J})\) a
diffeomorphism \(F : M \to M\) is a \textbf{CR diffeomorphism} if it preserves
the contact distribution and the \(J\)-operator. That is

\[
F_* \circ J = \tilde{J} \circ F_*.
\]

In terms of choices of coframes we are requiring

\[
F^* \tilde{\theta} = s \theta
\]

\[
F^* \tilde{\theta}^1 = \gamma \theta^1 + \delta \theta
\]

with \(s\) real, \(\gamma\) and \(\delta\) complex \(s \neq 0\), and \(\gamma \neq 0\).
Given two pseudo-hermitian structures, say $\{\theta, \theta^1\}$ and $\{\theta, \tilde{\theta}^1\}$ and a diffeomorphism $F : M^3 \to M^3$, we say that the two pseudo-hermitian structures are equivalent, and that $F$ is a **pseudo-hermitian diffeomorphism** if

$$F^* (\theta) = \theta$$

and

$$F^* (\tilde{\theta}^1) = \gamma \theta^1 + \delta \theta.$$
The **standard CR structure** on the three sphere $S^3$ is the one it inherits as a submanifold of $\mathbb{C}^2$.

$$H = TS^3 \cap JTS^3.$$  

$H$ is called the **standard contact distribution**. Restricting $\theta_0 = -i(\bar{z}dz + \bar{w}dw)$ to $S^3$ gives the **standard pseudo-hermitian structure**.
The induced CR structure on $S^3$ can be given by

\begin{align*}
\theta_0 &= -i(\bar{z}dz + \bar{w}dw) \\
\theta_0^1 &= wdz - zdw
\end{align*}

where these forms are restricted to $S^3$.

Note

\begin{align*}
d\theta_0 &= i\theta_0^1 \land \theta_0^1 \\
d\theta_0^1 &= \theta_0^1 \land \omega \\
\omega &= -2i\theta_0 \\
d\omega &= 2\theta_0^1 \land \theta_0^1
\end{align*}

“ The pseudo-hermitian curvature equals two.”
The Webster Connection

Theorem

Let \((\theta, \theta^1)\) be a pseudo-hermitian coframe. There exist unique functions \(R, A,\) and \(V,\) and an unique one-form \(\omega,\) so that

\[
d\theta = i\theta^1 \wedge \theta^1
\]
\[
d\theta^1 = \theta^1 \wedge \omega + A\theta \wedge \theta^1
\]
\[
\omega = -\overline{\omega}
\]
\[
d\omega = R\theta^1 \wedge \theta^1 + 2i\Im(V\theta^1) \wedge \theta.
\]

Further, if \(\theta^1\) is replaced by \(\theta^1 = \lambda \theta^1, \ |\lambda| = 1,\) then

\[
R = R, \quad A = \lambda^2 A, \quad V = \lambda V, \quad \omega = \omega - \lambda^{-1} d\lambda.
\]

Recall from previous slide that \(R = 2, \ A = 0, \ V = 0\) for the standard pseudo-hermitian structure.
Let $\phi$ and $\phi_1$ be one-forms with $\phi$ real giving the CR structure:

1. $\phi^\perp = H$,
2. $J\phi_1 = i\phi_1$,
3. $d\phi = i\phi_1\phi_1$.

**Theorem**

There exist unique one-forms $\phi_2$, $\phi_3$, $\phi_4$ and unique functions $R_C$ and $S$ such that

1. $\phi_2$ is imaginary and $\phi_4$ is real,
2. $d\phi_1 = -\phi_1\phi_2 - \phi\phi_3$,
3. $d\phi_2 = 2i\phi_1\phi_3 + i\phi_1\phi_3 - \phi\phi_4$,
4. $d\phi_3 = -\phi_1\phi_4 - \phi_2\phi_3 - R_C\phi\phi_1$,
5. $d\phi_4 = i\phi_3\phi_3 + (S\phi_1 + \overline{S\phi_1})\phi$. 

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If we replace $\phi$ by $\psi = |\nu|^2 \phi$ and $\phi_1$ by $\psi_1 = \nu \phi_1$ with a constant $\nu$ then the forms

$$\psi_2 = \phi_2, \quad \psi_3 = \frac{1}{\nu} \phi_3, \quad \psi_4 = \frac{1}{|\nu|^2} \phi_4$$

satisfy the equations in the Theorem with $R$ and $S$ replaced by

$$R = \frac{R}{|\nu|^2 \nu^2} \quad \text{and} \quad S = \frac{S}{|\nu|^2 \nu}.$$

$R$ and $S$ are relative invariants.
We want to choose a multiple of $\phi$ and a corresponding multiple of $\phi_1$ so that $R(x) \equiv 1$.

**Corollary**

If $R(p) \neq 0$, there are precisely two choices of $(\phi, \phi_1)$ such that in a neighborhood of $p$

1. $(\phi, \phi_1)$ give the CR structure,
2. $d\phi = i\phi_1\overline{\phi_1}$, and
3. $R \equiv 1$.

The eight parameter space of choices is cut down to only two choices.
If we denote one choice by \((\omega, \omega_1)\), then the other choice is\((\omega, -\omega_1)\).
We set \(\phi = \omega\) and \(\phi_1 = \omega_1\) and apply the theorem to obtain \(\phi_2\), \(\phi_3\), and \(\phi_4\).

\[\phi'_2 = \phi_2, \quad \phi'_3 = -\phi_3, \quad \text{and} \quad \phi'_4 = \phi_4.\]
So the curvature $R$ is a pseudo-hermitian invariant and the torsion $A$ is a relative invariant. NEED CLARIFY THIS
The group structure

\[ SU(2) = \left\{ \left( \begin{array}{cc} \alpha & -\beta \\ \beta & \bar{\alpha} \end{array} \right) : |\alpha|^2 + |\beta|^2 = 1 \right\} \subset O(4). \]

Identify \( SU(2) \) with \( S^3 \). \( SU(2) \) acts on \( S^3 \) on the left.

Facts

- Every left-invariant 2-plane distribution is contact.
- The standard contact distribution is left-invariant.
Start by describing all left-invariant CR structures with the standard contact distribution.

\[ B \subset \mathbb{C} \otimes H \subset \mathbb{C} \otimes TS^3. \]

\[ B = \{ Z \in \mathbb{C} \otimes TS^3 \mid \theta_0, (\alpha \theta^1_0 + \beta \bar{\theta}^1_0) Z = 0 \}. \]

Assume \( \alpha \neq 0, \ |\beta/\alpha| \neq 1. \)

\[ B = \{ \theta_0, \theta^1_0 + \mu \bar{\theta}^1_0 \}^\perp \]

satisfies \( B \cap \overline{B} = \{ 0 \}. \)
Let $\theta(\mu) = \theta_0^1 + \mu \theta_0^1$.

**Theorem**

1. The left-invariant CR structures $(\theta_0, \theta_1^1(\mu))$ and $(\theta_0, \theta_1^1(\mu'))$ are equivalent if and only if either $|\mu| = |\mu'|$ or $|\mu| = |\mu'|^{-1}$.

2. The left-invariant pseudo-hermitian structures $(a\theta_0, \theta_1^1(\mu))$ and $(a'\theta_0, \theta_1^1(\mu'))$ are equivalent if and only if $a = a'$ and $|\mu| = |\mu'|$.

3. Any left-invariant CR structure of pseudo-hermitian structure is equivalent to one with the standard contact distribution.

The third statement may be proved by group theory, but analysis yields more information.
We need the Webster and Cartan curvatures for $(\theta_0, \theta^1(\mu))$.

\[ d\theta = i\theta^1 \wedge \bar{\theta}^1 \]

\[ d\theta^1 = \theta^1 \wedge \left( -2i \left( \frac{1 + |\mu|^2}{1 - |\mu|^2} \right) \theta - \frac{4i\mu}{1 - |\mu|^2} \theta \wedge \bar{\theta}^1. \right) \]
We need the Webster and Cartan curvatures for \((\theta_0, \theta^1(\mu))\).

\[
d\theta = i\theta^1 \wedge \overline{\theta^1}
\]

\[
d\theta^1 = \theta^1 \wedge \left(-2i \left(\frac{1 + |\mu|^2}{1 - |\mu|^2}\right)\right)\theta - \frac{4i\mu}{1 - |\mu|^2} \theta \wedge \overline{\theta^1}.
\]

So

\[
A = -\frac{4i\mu}{1 - |\mu|^2}
\]

and

\[
R = 2 \left(\frac{1 + |\mu|^2}{1 - |\mu|^2}\right).
\]

Note that \(RA \neq 0\) for \(\mu \neq 0\).