A NON-HAUSDORFF SPACE OF CR EQUIVALENCE CLASSES

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ABSTRACT. Examples show that in three dimensions the natural topology on the space of equivalence classes of CR structures need not be Hausdorff. However, the space of equivalence classes of compact aspherical CR structures is Hausdorff. Thus the non-Hausdorff property is due to spherical or Levi-flat points.

1. DEFINITIONS

Let M be a given three dimensional manifold. A CR structure on M consists of a two-plane distribution H and a map $J : H \to H$ with $J^2 = -I$. Equivalently a CR structure may be thought of as a complex line bundle $V \subset \mathbb{C} \otimes TM$ with $\mathfrak{R}L$ and $\mathfrak{I}L$ linearly independent for each nowhere zero local section of V at every point of M. We write $V = [L]$ for the CR structure defined by such a vector field L. So a CR structure may be thought of as a map

$$M \to \mathcal{P}(\mathbb{C} \otimes TM)$$

of M into the projectivized complex tangent space (with a pointwise restriction on the image). This leads to a topology on the set of CR structures as the space of C^∞ maps from one manifold to another.

There is an extensive literature about such structures and the extensions to higher dimensions. See for instance [1], [5], [9].

The CR structure is said to be non-degenerate at a point if the two-plane distribution H is contact at that point. Otherwise it is said to be Levi-flat at that point. In terms of a non-zero section, Levi-flat at a point is a condition on the vector bracket of the real and imaginary components of the section. Namely,

$$[\mathfrak{R}L, \mathfrak{I}L] \in span\{\mathfrak{R}L, \mathfrak{I}L\}$$

at that point. The CR structure is said to be non-degenerate, respectively Levi-flat, if it is non-degenerate, respectively Levi-flat, at all points of M. Non-degenerate structures are also called strictly pseudoconvex.

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We may choose a frame for a non-degenerate CR structure and compute a function R called the Cartan curvature. This function is a relative invariant of the CR structure. Its value at a point depends on our choice of frame, but whether it is zero or nonzero does not depend on the choice. For the standard CR structure on S^3, namely the one induced by the complex structure on $\mathbb{C}^2$, we have $R \equiv 0$, so any point on M with $R = 0$ for some choice of frame, and thus for every choice of frame, is called a spherical point. A CR structure is aspherical if it has no spherical or Levi-flat points.

Two structures (H, J) and (H', J') are CR equivalent if there exists a diffeomorphism ϕ of M to itself such that

$$\phi_* H = H'$$

and

$$\phi_* J = J' \phi_*.$$

In terms of the complex line bundles, this becomes

$$\phi_* V = V'.$$

There is a natural topology on any space of equivalence classes. Let X be a topological space, $\mathcal{R} \subset X \times X$ an equivalence relation, and $\pi : X \to X/\mathcal{R}$ the projection taking $x \in X$ to its equivalence class. The topology on $X/\mathcal{R}$ is the strongest topology in which π is continuous; the set $U \subset X/\mathcal{R}$ is open if and only if $\pi^{-1}(U)$ is open in X. It is common for this topology to be non-Hausdorff. Here is perhaps the simplest example. Let the group $\mathbf{R}^*$ of nonzero real numbers act on the real line. Two real numbers x and y are equivalent, that is, $(x, y) \in \mathcal{R}$, if there exists $g \in \mathbf{R}^*$ such that $gx = y$. Thus there are two equivalence classes, say $[0]$ and $[1]$. The point $[1]$ is open, the point $[0]$ is not. Thus the topology is non-Hausdorff.

More generally, let a group G act on a topological space X and define the relation

$$\mathcal{R} = \{(x, y) \ : \ \exists g \in G, gx = y\}.$$

If G is the group of smooth diffeomorphisms acting on M and X is the set of CR structures on M, then $X/\mathcal{R}$ is the set of CR equivalence classes.

Theorem 1. *The space of CR equivalence classes is non-Hausdorff for any open and orientable 3-dimensional manifold.*

It is natural to conjecture that this is true for all 3-dimensional manifolds. In higher dimension there are involutivity conditions that might, in general, prevent constructions similar to that used in the proof below.

Theorem 2. *The space of CR equivalence classes of aspherical CR structures is Hausdorff for any compact 3-dimensional manifold.*

Let us say that an element x of a topological space X is a non-Hausdorff progenitor with respect to a relation $\mathcal{R}$ if there exists an element y, different from x, such that any open set in $X/\mathcal{R}$ containing $\pi(x)$ also contains $\pi(y)$. So Theorem 2 may be rephrased as:

Theorem 3. *Any non-Hausdorff progenitor for the CR structures on a 3-dimensional compact manifold must contain a Levi-flat point or a spherical point.*

There is a curious analogy with a result of Chakrabarti and Shaw [4]. They give an example of a Stein domain whose L^2 cohomology is non-Hausdorff. The boundary of the domain is Levi flat.

2. Examples

1- **A Levi-flat non-Hausdorff progenitor**

Let $M = \mathbb{R}^3$ with coordinates (x, y, u) and for each $t \in \mathbb{R}$ consider the complex vector field

$$L_t = \partial_{\overline{z}} - izt\partial_u$$

and the CR structure $V_t = [L_t]$ that it determines. Clearly, the sequence V_t converges to V_0 in the C^∞ topology. Note that V_0 is Levi-flat but V_t is not, $t \neq 0$.

Now let $t \neq 0$ and let $\phi : M \to M$ be the diffeomorphism

$$\phi(z, u) = (tz, tu).$$

Since

$$\begin{aligned} \phi_* L_t &= t(\partial_{\overline{z}} - iz\partial_u) \\ &= tL_1 \end{aligned}$$

we have that each V_t, $t \neq 0$, is CR equivalent to V_1. If π is the map of a CR structure to its equivalence class, then V_t is a smooth curve of CR structures with $\pi(V_t) = \pi(V_1)$ for all $t \neq 0$ but $\pi(V_0) \neq \pi(V_1)$. Thus the distinct points $\pi(V_1)$ and $\pi(V_0)$ in $X/\mathcal{R}$ cannot be separated by open sets.

Remark: Let $1 \leq m \leq n$. There exists a non-Hausdorff progenitor for the CR structures on $\mathbb{R}^{2n+1}$ for which the Levi form at each point has precisely m zero eigenvalues. To see this, just consider

$$\begin{aligned} L_j &= \partial_{\overline{z}_j} - iz_j t\partial u \qquad & j &= 1, \ldots, m \\ L_j &= \partial_{\overline{z}_j} - iz_j\partial u \qquad & j &= m+1, \ldots, n. \end{aligned}$$

Further, if M^{2n+1} is an open, parallelizable manifold then M^{2n+1} immerses into $\mathbb{R}^{2n+1}$ [6] and so has the same type of non-Hausdorff progenitor.

2- **A spherical non-Hausdorff progenitor**

The complex structure on $\mathbb{C}^2$ induces a CR structure on the hyperquadric

$$Q = \{(z,w) : \Im w = |z|^2\}.$$

This structure is locally equivalent to that induced on S^3 and so is spherical. It is easy to see that this is a non-Hausdorff progenitor. For this, let

$$M_t = \{(z,w) : \Im w = |z|^2 + t(z^2 + \overline{z}^2)|z|^4\}$$

and

$$\begin{array}{rl} F_t : \mathbb{R}^3 & \to M_t \\ F_t(z,u) & = (z, u + i(|z|^2 + t(z^2 + \overline{z}^2)|z|^4). \end{array}$$

For $t = 0$, the induced structure on $\mathbb{R}^3$ corresponds to the usual spherical structure on Q. We note that for each $t \neq 0$ there exists a biholomorphism Φ_t on $\mathbb{C}^2$ such that

$$\Phi_t : M_t \to M_1,$$

namely,

$$(z,w) \to (t^{\frac{1}{6}} z, t^{\frac{1}{3}} w).$$

So, for $t \neq 0$, the induced structures on $\mathbb{R}^3$ are all equivalent and are known to be aspherical, see for instance [5]. Thus the usual spherical structure is a non-Hausdorff progenitor.

3- **Lie group examples**

Let G be a three-dimensional Lie group with a CR structure given by a left invariant vector field L. So L may be thought of as an element of the complex Lie algebra $\mathfrak{G}$ and the CR structure generated by L as a point in the projectivized Lie algebra, $[L] \in \mathcal{P}(\mathfrak{G})$. The orbit of $[L]$ under the adjoint action of G on $\mathfrak{G}$ consists of CR structures equivalent to $[L]$. It may happen that some orbit is open with a boundary point inequivalent to $[L]$. Thus the induced topology on the space of equivalence classes in non-Hausdorff. By Theorem 2, the boundary point must define a Levi-flat or spherical CR structure. This is the case for the group $E(2)$ of Euclidean motions in the plane and for the Heisenberg group Q [2]. In fact, both equivalence spaces are two points $\{p,q\}$ with p open and q not open. For $E(2)$, p is aspherical and q is Levi-flat while for Q, p is spherical and q is Levi-flat.

3. Proofs

Either of the first two examples establishes Theorem 1 for $\mathbb{R}^3$. Since any open and orientable 3-manifold immerses in $\mathbb{R}^3$ [10], these examples may be pulled back to M.

We start the proof of the second theorem by recalling a result from general topology, see [8, p. 98]:

Let X be a topological space, $\mathfrak{R} \subset X \times X$ an equivalence relation, and $\pi : X \to X/\mathfrak{R}$ the map to equivalence classes where this latter space has the strongest topology in which π is continuous; $U \subset X/\mathfrak{R}$ is open if $\pi^{-1}(U) \subset X$ is open. If $\mathfrak{R}$ is a closed subset and π is an open map, then $X/\mathfrak{R}$ is Hausdorff.

We want to apply this when X is the space of aspherical C^∞ CR structures on a manifold M and $\mathfrak{R}$ is the relation of CR equivalence. Thus $\pi : X \to X/\mathfrak{R}$ is the map of an aspherical CR structure to the set of all its images under the group of smooth diffeomorphisms of M to itself. Denote this group by $\mathfrak{D}$. So

$$\pi : X \to X/\mathfrak{D}.$$

Lemma 1. *The map π is an open map.*

Lemma 2. *The set $\mathfrak{R}$ is closed.*

Proof of Lemma 1. Let $\mathcal{U}$ be an open set in X. We need to show that $\pi^{-1}(\pi(\mathcal{U}))$ is also open. $\mathfrak{D}$ acts on X by

$$\mathfrak{D} \times X \to X$$
$$(\phi, (H, J)) \to (\phi_* H, \phi_* J \phi_*^{-1}).$$

Since $\mathcal{U}$ is an open set so is $\mathfrak{D}(\mathcal{U})$. But

$$\mathfrak{D}(\mathcal{U}) = \pi^{-1}(\pi(\mathcal{U}))$$

and we are done. □

This first lemma holds just as well when X is the space of CR structures rather than the space of aspherical CR structures. So any subtlety must lie in the proof of our second lemma.

Proof of Lemma 2. We first need to briefly review what makes aspherical points special. We start with a non-degenerate CR structure (H, J) and, working near some point, find a real one-form ω and a complex one-form ω^1 such that

$$\omega(H) = 0,$$
$$\omega^1(\xi + iJ\xi) = 0, \quad \xi \in H \tag{3.1}$$
$$d\omega = i\omega^1 \wedge \overline{\omega^1} \quad mod\ \omega$$

The set of choices for these frames is a four-dimensional space. There is an algorithm which provides, in an almost unique way, a choice of these one-forms in a neighborhood of an aspherical point.

> Let $p \in M$ be an aspherical point of a CR structure V. There exist a neighborhood of p and a canonical choice of forms ω and ω^1 as in (3.1) such that ω is unique and ω^1 is unique up to sign. If $\phi : M \to M'$ is a CR diffeomorphism of $V \to V'$ and ω' and $\omega^{1\prime}$ are the corresponding canonical choice for V' at $\phi(p)$ then
>
> $$\phi^*(\omega') = \omega$$
> $$\phi^*(\omega^{1\prime}) = \pm\omega^1.$$

This algorithm is due to Cartan [3]. See also [7, p. 151]. We will call these one-forms our basic forms.

We are now ready to prove Lemma 2. Let $\mathfrak{R} \subset X \times X$ be the CR equivalence relation and let the points $(x_k, y_k) \in \mathfrak{R}$ converge to some point $(x, y) \in X \times X$. So we have

$$\pi(x_k) = \pi(y_k)$$

and we need to show that

$$\pi(x) = \pi(y).$$

Now π is continuous by definition (of the quotient topology) but we do not know (yet) that the quotient topology is Hausdorff. So convergent sequences need not have unique limits.

We may rewrite the convergence for $\{x_k\}$ as

$$(H_k, J_k) \to (H, K)$$

and for $\{y_k\}$ as

$$(H'_k, J'_k) \to (H', J')$$

For the basic forms we have

$$\omega_k \to \omega \text{ and } \omega'_k \to \omega'.$$

Because x_n and y_n are CR equivalent, we also have a sequence

$$\phi_n : x_n \to y_n$$

of CR diffeomorphisms. We claim that there is subsequence that converges along with all its derivatives and establishes the CR equivalence of x and y via a C^∞ diffeomorphism.

Lemma 3. *Let M and M' be three-dimensional compact and aspherical CR structures. There exist constants c_k, $k = 1, 2, \ldots$, depending only on these CR structures such that for any CR diffeomorphism of nearby CR structures*

$$\Phi : M \to M'$$

the differential satisfies

$$|D\Phi|_{C^k} < c_k.$$

Let us now assume this lemma. Thus $\{\phi_n\}$ is a bounded and equicontinuous family of smooth functions on a compact manifold and hence has a convergent subsequence,

$$\phi_{n_j} \to \phi$$

The sequence $\{D\phi_{n_j}\}$ is also equicontinuous and so $\phi \in C^1(M)$. The basic forms ω_k and ω'_k converge to ω and θ and so ϕ satisfies

$$\phi^* \omega' = \omega \text{ and } \phi^* \omega^{1\prime} = \omega^1. \tag{3.2}$$

Hence the limit ϕ is a CR diffeomorphism and thus $(x, y) \in \mathcal{R}$. This proves that $\mathcal{R}$ is closed.

Proof of Lemma 3. It is enough to work in a single coordinate patch. Since ω, ω^1, and $\overline{\omega^1}$ are independent, as are ω', $\omega^{1\prime}$ and $\overline{\omega^{1\prime}}$, the basic relation (3.2) may be rewritten as

$$\frac{\partial \phi^j}{\partial x_k} = A^j_k(\phi(x))$$

with A^j_k bounded by constants derived from the basic forms and valid for nearby CR structures. This establishes the desired C^0 estimate for $D\phi$ and shows how to derive the estimates for all the higher derivatives. □

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