Left Invariant CR Structures on S^3

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  - Curvature and torsion

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- Conjugate CR structures

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Wanted simple examples of pseudo-hermitian structures with torsion.

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Then J is completely determined by the eigenspace corresponding to the eigenvalue i (or to the eigenvalue -i). So a CR structure is just as well given by a complex line bundle $B \subset \mathbf{C} \otimes H$,

$$B \cap \overline{B} = \{0\}.$$

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There exists some θ^1 such that

1. $d\theta = i\theta^1 \wedge \overline{\theta^1}$ (or $d(-\theta) = i\theta^1 \wedge \overline{\theta^1}$)
2. $X \in H \implies \theta^1(X + iJX) = 0$. (Equivalently, $J\theta^1 = i\theta^1$)

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The Standard Structures

The **standard CR structure** on the three sphere S^3 is the one it inherits as a submanifold of $\mathbf{C}^2$.

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$$H = TS^3 \cap JTS^3.$$

H is called the **standard contact distribution**.
Choosing $\theta_0 = -i(\overline{z}dz + \overline{w}dw)$ give the **standard pseudo-hermitian structure** .

The natural choice for a coframe for these structures is

$$\{\theta_0, \theta_0^1\}$$

with

$$\begin{aligned} \theta_0 &= -i(\overline{z}dz + \overline{w}dw) \\ \theta_0^1 &= wdz - zdw \end{aligned}$$

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where these forms are restricted to S^3.
Note for later that

$$\begin{aligned} d\theta_0 &= i\theta_0^1 \wedge \overline{\theta_0^1} \\ d\theta_0^1 &= \theta_0^1 \wedge \omega \\ \omega &= -2i\theta_0. \end{aligned}$$

Given two CR structures (M, H, J) and $(M, \tilde{H}, \tilde{J})$ a diffeomorphism $F : M \to M$ is a **CR diffeomorphism** if it preserves the two-plane distribution and the J-operator. That is

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In terms of choices of coframes we are requiring

$$F^* \tilde{\theta} = s\theta$$

$$F^* \tilde{\theta^1} = \gamma\theta^1 + \delta\theta$$

with s real, γ and δ complex $s \neq 0$, and $\gamma \neq 0$.

Given two pseudo-hermitian structures, say $\{\theta, \theta^1\}$ and $\{\theta, \tilde{\theta^1}\}$ and a diffeomorphism $F : M^3 \to M^3$, we say that the two pseudo-hermitian structures are equivalent, and that F is a **pseudo-hermitian diffeomorphism** if

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$$F^*(\theta) = \theta$$

and

$$F^*(\tilde{\theta^1}) = \gamma\theta^1 + \delta\theta.$$

The Webster Connection

Theorem

Let (θ, θ^1) be a pseudo-hermitian coframe. There exist unique functions R, A, and V, and an unique one-form ω, so that

$$\begin{aligned} d\theta &= i\theta^1 \wedge \overline{\theta^1} \\ d\theta^1 &= \theta^1 \wedge \omega + A\theta \wedge \overline{\theta^1} \\ \omega &= -\overline{\omega} \\ d\omega &= R\theta^1 \wedge \overline{\theta^1} + 2i\Im(V\overline{\theta^1}) \wedge \theta. \end{aligned}$$

Further, if θ^1 is replaced by $\boldsymbol{\theta^1} = \lambda\theta^1$, $|\lambda| = 1$, then

$$\mathbf{R} = R, \quad \mathbf{A} = \lambda^2 A, \quad \mathbf{V} = \lambda V, \quad \boldsymbol{\omega} = \omega - \lambda^{-1} d\lambda.$$

So the curvature R is a pseudo-hermitian invariant and the torsion A is a relative invariant.

The group structure

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$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \leftrightarrow (\alpha, \beta) \in C^2$$

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$$(0, 1, 0, 0), \quad (0, 0, 1, 0), \quad \text{and } (0, 0, 0, 1)$$

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$$\begin{aligned} L_1 &= (-b, a, -d, c)) \\ L_2 &= (-c, d, a, -b) \\ L_3 &= (-d, -c, b, a). \end{aligned}$$

Contact structures

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For any other left-invariant distribution we can choose a basis

$$\begin{aligned} U &= L_1 + uL_3 \\ V &= L_2 + vL_3 \end{aligned}$$

with real constants u and v.

Lemma

Each left-invariant 2-plane distribution on S^3 is a contact structure.

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$$2u^2 + 2v^2 = -2.$$

Lemma

If $\mathcal{D}$ is a left-invariant 2-plane distribution on S^3 then there is some $\Phi : S^3 \to S^3$ such that the induced map

$$\Phi_* TS^3 \to TS^3$$

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Restrict CR and pseudo-hermitian structures to have the standard distribution.

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$$\mu \text{ and } \frac{1}{\mu} \text{ define conjugate CR structures.}$$

For the pseudo-hermitian coframe

$$\begin{aligned} \theta &= \theta_0 \\ \theta^1 &= \lambda(\theta_0^1 + \mu\overline{\theta_0^1}) \end{aligned}$$

with

$$|\lambda|^2(1 - |\mu|^2) = 1.$$

we have

$$d\theta^1 = \theta^1 \wedge \left(-2i \left(\frac{1+|\mu|^2}{1-|\mu|^2} \right) \right) \theta - \frac{4i\mu}{1-|\mu|^2} \theta \wedge \overline{\theta^1}.$$

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Webster connection form

$$\omega = -2i\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right)\theta_0.$$

Torsion

$$A = -\frac{4i\mu}{1-|\mu|^2}$$

Curvature

$$R = 2\left(\frac{1+|\mu|^2}{1-|\mu|^2}\right).$$

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Theorem

The map

$$\{\mu \in C,\ |\mu| < 1\} \to \mathcal{S}$$

is surjective with fiber given by $|\mu|$.

Remark

The same result and proof hold for $|\mu| > 1$ *and* θ *replaced by* $-\theta$.

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$$F^*(\theta|_{(\tilde{z},\tilde{w})}) = \theta|_{(z,w)}$$

and

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This illustrates a general theorem:
A spc CR structure on M^3 with nonzero "CR curvature" admits at most two CR diffeomorphisms leaving a given point fixed.
If $\mu = 0$ the dimension of the isotropy group of a point is 5.

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together with $F^*(\tilde{\theta}^1) = \alpha\theta^1$ to derive

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Use

$$A = -\frac{4i\mu}{1 - |\mu|^2}$$

to conclude that $|\mu| = |\tilde{\mu}|$.

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Theorem

There exist unique one-forms ϕ_2, ϕ_3, ϕ_4 and unique functions $R(x)$ and $S(x)$ such that

1. *ϕ_2 is imaginary and ϕ_4 is real,*
2. $d\phi_1 = -\phi_1\phi_2 - \phi\phi_3$,
3. $d\phi_2 = 2i\phi_1\overline{\phi_3} + i\overline{\phi_1}\phi_3 - \phi\phi_4$,
4. $d\phi_3 = -\phi_1\phi_4 - \overline{\phi_2}\phi_3 - R\phi\overline{\phi_1}$,
5. $d\phi_4 = i\phi_3\overline{\phi_3} + (S\phi_1 + \overline{S\phi_1})\phi$.

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R and S are relative invariants.

$R(p) \neq 0$ implies that (M, H, J) is nonumbilic at p.

Corollary

A left invariant CR structure on S^3 with $\mu \neq 0$ has no umbilic points.

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Corollary

If $R(p) \neq 0$, there are precisely two choices of (ϕ, ϕ_1) such that in a neighborhood of p

1. (ϕ, ϕ_1) *give the CR structure,*
2. $d\phi = i\phi_1\overline{\phi_1}$, *and*
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We set $\phi = \omega$ and $\phi_1 = \omega_1$ and apply the theorem to obtain ϕ_2, ϕ_3, and ϕ_4.

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We set $\phi = \omega$ and $\phi_1 = \omega_1$ and apply the theorem to obtain ϕ_2, ϕ_3, and ϕ_4. If instead we had set $\phi' = \omega$ and $\phi_1' = -\omega_1$, we would have obtained

$$\phi_2' = \phi_2, \quad \phi_3' = -\phi_3, \text{ and } \phi_4' = \phi_4.$$

Theorem

If F is a CR diffeomorphism between left-invariant CR structures characterized by μ *and* $\tilde{\mu}$ *and with the standard contact distribution then either* $|\mu| = |\tilde{\mu}|$ *or* $|\mu| = 1/|\tilde{\mu}|$*.*