A CONJECTURE OF TRAUTMAN

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In 1998 the physicist Andre Trautman conjectured that a three-dimensional CR manifold is locally realizable if and only if its canonical bundle admits a closed nowhere zero section. First we review the relevant definitions and in the next section give the physical context. In Section 3 we outline the earlier results in [2] which had proved a weak version of the Conjecture.

A **CR structure** on a three-dimensional manifold M is a two-plane distribution $H \subset TM$ and a fiber preserving anti-involution $J : H \to H$. We denote this structure by (M, H, J). It is often useful to extend J by complex linearity to a map

$$J : \mathbf{C} \otimes H \to \mathbf{C} \otimes H.$$

Then J is completely determined by the eigenspace corresponding to the eigenvalue i (or to the eigenvalue -i).

An equivalent definition of a CR structure on a three-dimensional manifold may be given in terms of a complex line bundle: A CR structures on M is a line bundle $B \subset \mathbf{C} \otimes TM$ with the property that $B \cap \overline{B}$ contains only the zero section. Then

$$H = \{\Re Z \ : \ Z \in B\}$$

is of rank 2 and J is defined on $\mathbf{C} \otimes H = B \oplus \overline{B}$ by setting

$$J(Z) = iZ \text{ if } Z \in B$$

and

$$J(Z) = -iZ \text{ if } Z \in \overline{B}.$$

So for $X - iY \in B$

$$JX = Y \text{ and } JY = -X.$$

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Example Let $M^3 \subset \mathbf{C}^2$ be a real hypersurface and let J denote the usual operator on $\mathbf{R}^4$ giving the complex structure. Set $H_p = T_pM \cap JT_pM$ for each $p \in M$. Now J acts on H and (M,H,J) is a CR structure. Or, to use the alternative definition, just take

$$B = T^{1,0}(\mathbf{C}^2) \cap \mathbf{C} \otimes TM$$

where $T^{1,0}$ is the linear span of

$$\left\{ \frac{\partial}{\partial z_1}, \quad \frac{\partial}{\partial z_2} \right\}$$

(and $T^{0,1}$ is the span of the conjugates).

So later we write $B = T^{1,0}(M) = T^{1,0}$ and write $T^{0,1}$ for $\overline{B}$.

The canonical bundle is another complex line bundle associated to a CR structure. It is a subbundle of the second exterior product. For a real hypersurface in $\mathbf{C}^2$ it is generated by the two-form $dz_1 \wedge dz_2$ restricted to M. More generally, if the CR structure is given by a complex line bundle B then

$$\Omega = \{\omega \in \mathbf{C} \otimes \Lambda^2(TM) \ : \ i_b\omega = 0 \text{ for all } b \in \overline{B}\}.$$

The interior product $i_b\omega$ is given by $i_b\omega(X) = \omega(b \wedge X)$.

Definition. *(M,H,J) is realizable in a neighborhood of p if there exist complex functions f_1 and f_2 such that*

$$(X + iJX)f_k = 0$$

for all $X \in H$ and

$$F : M \to \mathbf{C}^2$$
$$x \to (\Re f_1, \Im f_1, \Re f_2, \Im f_2)$$

is an embedding.

It follows upon identifying M with its image $F(M)$ that the original structure (M, H, J) coincides with the CR structure induced as in the Example.

We digress briefly to discuss higher-dimensional CR structures and return to this in Section 3.

Definition. (M^{2n+1}, B) *is a CR manifold if $B \subset \mathbf{C} \otimes TM$ is a vector subspace of rank n with $B \cap \overline{B} = \{0\}$ and $[\Gamma B, \Gamma B] \subset \Gamma B$. I.e., the commutator of local sections of B is always in B.*

More precisely, we have defined a CR manifold of hypersurface type.

Definition. *(M^{2n+1}, B) is realizable if there is an embedding $F : M \to \mathbf{C}^{n+1}$ with, after identifying M with $F(M)$,*

$$T^{1,0}(\mathbf{C}^{n+1}) \cap \mathbf{C} \otimes TM = B.$$

The canonical bundle is now a complex line bundle in the exterior product $\Lambda^{n+1}(\mathbf{C} \otimes TM^{2n+1})$. Namely,

Definition. *The canonical bundle is*

$$\Omega = \{\omega \in \mathbf{C} \otimes \Lambda^{n+1}(TM) \;:\; i_v\omega = 0,\ \forall v \in T^{0,1}\}.$$

Definition. *A function $f : M \to \mathbf{C}$ is a CR function if $Lf = 0$ for all $L \in T^{0,1}$.*

Lemma 1.1. *M^{2n+1} is realizable in $\mathbf{C}^{n+1}$ if there exist CR functions $f_1, \ldots, f_{n+1}$ such that*

$$df_1 \wedge \ldots \wedge df_{n+1} \neq 0. \tag{1.1}$$

Proof. Let $L_1, \ldots L_n$ be a basis for $T^{0,1}$ and let T be any nonzero vector transverse to H. From (1.1) and using that the functions are CR, we have

$$df_1 \wedge \ldots \wedge df_{n+1}(\overline{L}_1, \ldots, \overline{L}_n, T) \neq 0.$$

So $df_jT \neq 0$ for some j, say $j = n+1$, which now implies

$$df_1 \wedge \ldots \wedge f_n \wedge df_{n+1} \wedge \overline{df_1} \wedge \ldots \wedge \overline{df_n} \neq 0.$$

Thus

$$F = (f_1, \ldots, f_{n+1})$$

is a local embedding. Indeed perhaps after multiplying F by i, $F(M)$ has the form

$$\Im z_{n+1} = f(z_1, \ldots, z_n, \Re z_{n+1})$$

□

The realizability problem is quite subtle. For instance, most three-dimensional C^∞ CR structures are not locally realizable [4], [7].

Most realizability results in higher dimensions concern strictly pseudo-convex CR structures.

Definition. *A CR structure (M, B) is strictly pseudo-convex if the quadratic form*

$$L \in B \to [L, \overline{L}] \quad mod\{B \oplus \overline{B}\}$$

is definite.

Such structures are realizable if $dim\ M \geq 7$. See [1] and [5] for the original proofs and [11] for a variation.

Although, as we said, the general realizability problem is subtle there are two easy results.

Proposition 1. *Real analytic CR manifolds are locally realizable.*

A proof can be found, for instance, in [3, page 22].

Proposition 2. *A CR manifold admiting a vector field v transverse to H and preserving the CR structure is locally realizable.*

To preserve the CR structure means that the Lie derivative in the direction of v satisfies

$$\pounds_v T^{1,0} = T^{1,0}$$

A generalization of this result is important in Section 3 and will be proved there.

2.

We first wish to explain the observation of [8] that a shear-free congruence of null geodesics on a four-dimensional manifold induces a three-dimensional CR structure on a quotient manifold.

Let M^4 be a Lorentz manifold with metric g and let k be a null vector field, $g(k,k) = 0$. Let K be the real line bundle generated by k. Set

$$K_p^\perp = \{v \in T_pM : g(v,k) = 0\}.$$

Note that $K \subset K^\perp$ and that $K^\perp/K$ is an R^2 bundle on M. Following the notation in [10], let $n \in K^\perp$. Denote the equivalency class of n in $K^\perp/K$ by $[n]$ and use the same notation for $n \in \mathbf{C} \otimes (K^\perp/K) = \mathbf{C} \otimes K^\perp/\mathbf{C} \otimes K$.

Lemma 2.1. *The metric g induces a well-defined positive definite inner product on $K^\perp/K$.*

Proof. Let $[n_1]$ and $[n_2]$ belong to the fiber of $K^\perp/K$ over some point of M. Define $g([n_1],[n_2])$ to be $g(n_1,n_2)$. If v_1 and v_2 are different choices then $v_j = n_j + a_j k$ and so

$$\begin{aligned} g(v_1,v_2) &= g(n_1 + a_1 k, n_2 + a_2 k) \\ &= g(n_1,n_2) \end{aligned}$$

since k is a null vector and $n_j \in K^\perp$. This shows that g is well-defined.

To see that g is definite, assume that for some $[n]$ we have

$$g([n],[n]) \equiv g(n,n) = 0.$$

By the definitions of k and $K^\perp$ we also have

$$g(k,k) = 0$$

and

$$g(k,n) = 0.$$

So either n is a multiple of k or g vanishes on a two-dimensional plane. The second alternative is not possible for a Lorentz metric. So $n = ak$ and thus $[n] = 0$. Hence g is definite, and since it arises from a Lorentz metric it is positive definite.

□

Fix an orientation for $K^\perp/K$ (this is not a problem, as long as we care only about local results) and then let $J : K^\perp/K \to K^\perp/K$ be the operation of rotation by $\pi/2$ radians with respect to the induced metric and orientation. Finally, set

$$N = \{n \in \mathbf{C} \otimes K^\perp : J[n] = -i[n]\}.$$

Note that N is a two-dimensional complex vector bundle on M. Extend the inner product g to N as a complex linear form. For $n_1 = \xi + iJ\xi$ and $n_2 = \eta + iJ\eta$ in N we have

$$\begin{aligned} g(n_1, n_2) &= g(\xi,\xi)) + ig(J\xi,\eta) + ig(\xi, J\eta) - g(J\xi, J\eta) \\ &= 0 \end{aligned}$$

since J is rotation by $\pi/2$ radians. So N is said to be totally null. On the other hand,

$$g(n_1, \overline{n_1}) = 2g(\xi,\xi) \neq 0.$$

We have

$$N \subset \mathbf{C} \otimes K^\perp \subset \mathbf{C} \otimes TM$$

and

$$N \cap \overline{N} = \mathbf{C} \otimes K, \qquad N + \overline{N} = \mathbf{C} \otimes K^\perp.$$

Now consider the flow generated by the vector field k. For small values of the time parameter, the orbit space is a three-dimensional manifold (again, for local results this is clear); call it M'. Without additional assumptions on k the bundle N does not project to a well-defined subbundle of $\mathbf{C} \otimes TM'$. Here is where physics enters.

We temporarily drop the assumption that k is null.

Definition. [8, page 1426] *The vector field k is said to be conformally geodesic if the associated flow preserves $K^\perp$ and $g(k,k)$ does not change sign.*

Note that this definition depends only on the conformal class of g and also that in Riemannian geometry the condition on the flow and $g(k,k)=c$ imply $\nabla_k k = 0$.

The flow condition may be rewritten as

$$\pounds_k K^\perp \subset K^\perp.$$

and is equivalent to

$$g(k) \wedge \pounds_k g(k) = 0 \tag{2.1}$$

where $g(k)$ is the one-form defined by $g(k)v = g(k,v)$. To see this equivalence, we first note that if v is a vector field satisfying $g(k)v = 0$ then also $k(g(k)v) = 0$ and so

$$(\pounds_k g(k))v + g(k)\pounds_k v = 0. \tag{2.2}$$

We want to derive $g(k) \wedge \pounds_k g(k) = 0$. It is enough to show that

$$g(k)v = 0 \implies \pounds_k g(k)v = 0.$$

That is, if $g(k)$ and $\pounds_k g(k)$ have the same kernel then these one-forms are linearly dependent. So assume $\pounds_k K^\perp \subset K^\perp$ and $g(k)v = 0$. We now have

$$g(k)v = 0 \Rightarrow v \in K^\perp \Rightarrow \pounds_k v \in K^\perp \Rightarrow g(k)\pounds_k v = 0 \Rightarrow \pounds_k g(k)v = 0$$

where the last implication follows from (2.2).

On the other hand, if $g(k) \wedge \pounds_k g(k) = 0$, then

$$g(k)v = 0 \Rightarrow (\pounds_k g(k)v = 0 \Rightarrow g(k)\pounds_k v = 0 \Rightarrow \pounds_k K^\perp \subset K^\perp.$$

We are interested in the case where k is a null vector, $g(k,k) = 0$. When k is null the foliation of M by the integral curves of k is called a congruence of null geodesics.

The Lorentz metric g induces a degenerate inner product on $K^\perp$ and therefore also a (degenerate) conformal structure.

Definition. *A conformally geodesic vector field is shear-free if the associated flow preserves the conformal structure of $K^\perp$.*

The physical hypothesis that k generates a shear-free congruence of null geodesics also can be formulated in terms of the Lie derivative.

Theorem 2.1. [9] *A vector field k on a manifold M^4 with Lorentz metric g generates a shear-free congruence of null geodesics if and only if*

$$g(k,k) = 0 \tag{2.3}$$

$$\pounds_k g = \lambda g + \phi \otimes g(k) \tag{2.4}$$

where λ is a function, ϕ is a one-form, $g(k)$ is as defined above, and $\phi \otimes g(k)$ signifies the symmetric product constructed from the one-forms.

Proof. We first show that (2.4), together with (2.3), implies (2.1) and hence k is a conformally geodesic vector field. We start with the Leibniz rule:

$$k(g(u,v)) = \pounds_k g(u,v) + g(\pounds_k u, v) + g(u, \pounds_k v).$$

Setting $u = k$ and rearranging this becomes

$$\pounds_k g(k,v) = k(g(k,v)) - g(k, \pounds_k v).$$

Further

$$k(g(k,v)) = k(g(k)v) = (\pounds_k g(k))v + g(k, \pounds_k v)$$

and so

$$(\pounds_k g)(k,v) = (\pounds_k g(k))(v).$$

Now

$$\begin{aligned}
(g(k) \wedge \pounds_k g(k))(u,v) &= (g(k)u)(\pounds_k g(k)v) - (g(k)v)(\pounds_k g(k)u) \\
&= (g(k)u)(\pounds_k g)(k,v) - (g(k)v)(\pounds_k g(k,u)) \\
&= g(k,u)(\lambda g(k,v) + \phi \otimes g(k)(k,v)) \\
&\qquad - g(k,v)(\lambda g(k,u) + \phi \otimes g(k)(k,u)) \\
&= 0.
\end{aligned}$$

We know that $g(k) \wedge \pounds_k g(k) = 0$ implies that $K^{\perp}$ is preserved and so k is conformally geodesic.

To show that the conformal class g induces on $K^{\perp}$ is constant along the flow on M induced by k, we let $v \in K^{\perp}$ be constant along the flow. So $g(k)v = 0$ and $\pounds_k v = 0$. Thus

$$\begin{aligned}
k(g(v,v)) &= \pounds g(v,v) \\
&= (\lambda g + \phi \otimes g(k))(v,v) \\
&= \lambda g(v,v).
\end{aligned}$$

This gives us an ordinary differential equation. If local coordinates (t,x) are introduced with $k = \partial_t$ then the equation has the form

$$\frac{\partial f(t,x)}{\partial t} = \lambda(t,x) f(t,x)$$

and the solutions are

$$f(t,x) = \Lambda(t,x) f(0,x)$$

for some function Λ. Thus

$$g(v,v)(t,x) = \Lambda(t,x) g(v,v)(0,x).$$

This shows that the conformal class of the metric on $K^\perp$ does not change under the flow.

Conversely,we want to show that if k generates a shear-free congruence of null geodesics then there exist a scalar function λ and a one-form ϕ satisfying (2.4). To see this, we start with a frame invariant along the orbits, labeled e_1, e_2, e_3, e_4 with $\{e_1, e_2, e_3\}$ a basis for $K^\perp$ and $g(e_4, e_4) = 0$. Let $0 \le i \le 3,\ 0 \le j \le 3$. Note that $g(k)e_i = 0$ and $g(k)x_4 \neq 0$. For $p \in M$ parametrize the orbit through p by t. Since the conformal class of g on $K^\perp$ is constant

$$g(e_i, e_j)|_t = \Lambda(t) g(e_i, e_j)|_p$$

Thus

$$(\pounds_k g)(e_i, e_j)|_p = (\pounds_k \Lambda) g(e_i, e_j)|_p.$$

Define

$$\begin{aligned} \lambda &= \pounds_k \Lambda \\ \phi(e_i) &= (g(k)e_4)^{-1}\Big((\pounds_k g)(e_i, e_4)) - \pounds_k \lambda g(e_i, e_4)\Big), \quad 0 \le i, j \le 4. \end{aligned}$$

We have for $0 \le i, j \le 3$

$$(\lambda g + \phi \otimes g(k))(e_i, e_j) = (\pounds_k \Lambda) g(e_i, e_j) = (\pounds_k g)(e_i, e_j),$$

while for $i \le 4$ we have

$$\begin{aligned} (\lambda g + \phi \otimes g(k))(e_i, e_4) &= (\pounds_k \Lambda) g(e_i, e_4) + \phi(e_i) g(k) e_4 \\ &= (\pounds_k \Lambda) g(e_i, e_4) + \pounds_k (g(e_i, e_4) - (\pounds_k \Lambda) g(e_i, e_4) \\ &= (\pounds_k g)(e_i, e_4). \end{aligned}$$

Thus

$$\lambda g + \phi \otimes g(k) = \pounds_k g.$$

□

Let π denote the map of M to the orbit space

$$\pi : M \to M'.$$

Lemma 2.2. *Under the conditions of the Theorem, $\pi_*(N)$ is a complex line bundle $\overline{B} \subset \mathbf{C} \otimes TM'$ which satisfies $B \cap \overline{B} = \{0\}$.*

Proof. Since $K^\perp$ is itself invariant under the flow, $K^\perp/K$ projects to a well-defined two-plane distribution H on M' and on H we have a well-defined conformal class of metrics. Thus $\mathbf{C} \otimes TH$ splits into the eigenspaces of J

$$\mathbf{C} \otimes TH = B \oplus \overline{B}.$$

with $\pi_* N = \overline{B}$. □

That is, the physical assumptions lead to a CR structure on the orbit space. Further, as we now show, the same conditions provide a two-form F associated to N which itself also passes down to M'. The interest in such a two-form comes from considerations of Maxwell's equations. In classical physics, the components of the magnetic and electrical fields can be used to construct a real two-form F, called the Faraday tensor. Then, in the absence of charge, Maxwell's equations become $dF = 0$. Naturally, in relativistic physics the situation is more complicated.

To define F we first find a basis for N. Let $\xi \in K^\perp$ and $\xi \notin K$.Choose any $\eta \in K^\perp$ such that $J[\xi] = [\eta]$. Then $n = \xi + i\eta$ and k form a basis for N.

Let $g(k)$, defined above, and $g(n)$, defined in the same way, be one-forms on M. Set

$$F = g(n) \wedge g(k).$$

Note that F is nowhere zero since the one-forms $g(n)$ and $g(k)$ are independent. For example, $g(n)\overline{n} \neq 0$ while $g(k)\overline{n} = 0$.

The two-form F is associated to N in the following sense:

Lemma 2.3. $N = \{v \in \mathbf{C} \otimes TM \ : \ i_v F = 0\}$.

Proof. We have $g(k,k) = 0$ because k is null; $g(k,n) = 0$ because $N \subset \mathbf{C} \otimes K^\perp$; and $g(n,n) = 0$ because N is totally null. So for our basis $i_k F = 0$ and $i_n F = 0$. Thus

$$N \subset \{v \in \mathbf{C} \otimes TM \ : \ i_v F = 0\}.$$

Now let $t \in \{v \in \mathbf{C} \otimes TM \ : \ i_v F = 0\}$. So

$$g(n,t)g(k) - g(k,t)g(n) = 0.$$

The independence of $g(n)$ and $g(k)$ implies $t \in \mathbf{C} \otimes K^\perp$ at some point of M. Thus

$$t = \alpha n + \beta \overline{n} + \gamma k$$

for constants α, β, γ. Since $g(n,t) = 0$ and $g(n, \overline{n}) \neq 0$, we see that $\beta = 0$ and thus $t \in N$. □

We may use F to define a two-form on M': Let t_1 and t_2 be vectors in $\mathbf{C} \otimes \mathbf{TM'}$. Lift t_j to a vector $t_j + \alpha_j k$ in $\mathbf{C} \otimes TM$. Then

$$\begin{aligned} F((t_1 + \alpha_1 k) \wedge (t_2 + \alpha_2 k)) &= g(n, t_1 + \alpha_1 k) g(k, t_2 + \alpha_2 k) \\ &- g(n, t_2 + \alpha_2 k) g(k, t_1 + \alpha_1 k) \\ &= g(n) \wedge g(k)(t_1 \wedge t_2). \end{aligned}$$

So F evaluated on the lift is independent of choices and gives a well-defined two-form on M'. Call this form F'. For $t \in \overline{B} = \mathbf{C} \otimes T^{0,1}(M')$ the natural lift, also called t is in N. Thus from the Lemma

$$t \in \mathbf{C} \otimes T^{0,1}(M') \implies i_t F' = 0.$$

Hence F' is section of the canonical bundle of M' and is nowhere zero.

In summary, the local quotient of a Lorentzian manifold under a shear-free congruence of null geodesics is a CR manifold which has a nowhere zero section of its canonical bundle. This section being closed is related to Maxwell's equationa and so is a reasonable hypothesis for physicists. We now repeat Trautman's conjecture.

Conjecture 2.1. *If a CR manifold M^3 admits a nowhere zero closed section of its canonical bundle, then the CR structure is locally realizable.*

As we have seen, the converse is true even globally.

3.

A weak version of the conjecture is true and holds for all dimensions.

Functions satisfying

$$df_1 \wedge \ldots \wedge df_k \neq 0$$

are called independent. Functions satisfying

$$df_1 \wedge \ldots \wedge df_k \wedge d\overline{f_1} \wedge \ldots \wedge d\overline{f_k} \neq 0$$

are called strongly independent.

Example. *The hyperquadric $Q^3 \subset \mathbf{C}^2$ is defined by $\Im z_2 = |z_1|^2$. The bundle $T^{0,1}$ is generated by*

$$L = \partial_{\overline{z_1}} - iz\partial_u$$

where $u = \Re z_2$. The CR function $f = z$ is strongly independent; The function $f = u + i|z|^2$ is independent, but not strongly independent (at the origin).

The following theorem preceded the formulation of Trautman's Conjecture and establishes a weak form.

Theorem 3.1. [2] *If the CR structure M^{2n+1} has n strongly independent CR functions near p and if the canonical bundle has a closed nowhere zero section then M^{2n+1} is realizable in a neighborhood of p.*

The proof depends on the following complex version of Proposition 2

Proposition 3. *M is realizable in a neighborhood of p if and only if there exists a complex vector field Y near p such that*

- *Y is transverse to $T^{1,0} \oplus T^{0,1}$*
- *$\pounds_Y T^{1,0} = T^{1,0}$.*

Thus the existence of a real vector field such that $\pounds_v T^{1,0} = T^{1,0}$ is very special (since most realizable CR structures do not have such a vector field) but the existence of such a complex vector field characterizes realizability.

Proof. We first prove the necessity. So assume M is realizable near p. Without loss of generality we assume $p = 0$ and M is given as

$$M = \{(z_1, \dots, z_{n+1}) : \Im z_{n+1} = \rho(z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_{n-1}, \Re z_{n+1}\}.$$

Define $\overline{Y}$ by

$$dz_{n+1}(\overline{Y}) = 1, \qquad dz_j(\overline{Y}) = d\overline{z}_j(\overline{Y}) = 0, \quad 1 \le j \le n. \tag{3.1}$$

Note that $\overline{Y}$ (and also Y) is transverse to $T^{1,0} \oplus T^{0,1}$. Set

$$\omega = dz_1 \wedge \dots \wedge dz_{n+1}|_M.$$

This is a nowhere zero closed section of the canonical bundle. As a consequence of Cartan's formula

$$\pounds_v = di_v + i_v d$$

we have

$$\pounds_{\overline{Y}} \omega = d(i_{\overline{Y}} \omega) + i_{\overline{Y}} d\omega = 0.$$

This implies $\pounds_{\overline{Y}} T^{0,1} = T^{0,1}$ and so also

$$\pounds_Y T^{1,0} = T^{1,0}.$$

Conversely, we will assume that $\pounds_Y T^{1,0} = T^{1,0}$ with Y transverse to $T^{1,0} \oplus T^{0,1}$, and show that M is locally realizable. This is just a slight modification of a standard proof of Proposition 2. Extend Y and each of the vectors in $T^{1,0}$ to $\mathbf{C} \otimes T(M \times \mathbf{R})$ by taking them constant in the $\mathbf{R}$ direction. Let Y still denote this extension and let V denote the extension of the bundle $T^{1,0}$. Set Z to be the complex line bundle spanned by $Y + i\frac{\partial}{\partial t}$ where t is the natural parameter for $\mathbf{R}$. Then $W = V \oplus Z$ satisfies

$$W \cap \overline{W} = \{0\} \text{ and } W + \overline{W} = \mathbf{C} \otimes T(M \times \mathbf{R}).$$

Finally, as is easily seen, W is closed under the commutation of vector fields,

$$[\Gamma W, \Gamma W] \subset \Gamma W.$$

Thus W satisfies the conditions of the Newlander-Nirenberg Theorem [6] and so defines a complex structure on $M \times \mathbf{R}$. Since $W \cap \mathbf{C} \otimes TM =$

$T^{1,0}(M \times R)$, the CR structure induced on M is the one we started with. □

All that is left to do in the proof of Theorem 3.1 is to show that if $f_1, \ldots, f_n$ are CR functions on M^{n+1} with

$$df_1 \wedge \ldots \wedge d\overline{f}_n \neq 0$$

and if ω is a nowhere zero section of the canonical bundle with

$$d\omega = 0$$

then there is a complex vector field Y with

- Y transverse to $T^{1,0} \oplus T^{0,1}$
- $£_Y T^{1,0} = T^{1,0}$.

We just use the closed section to find a replacement for dz_{n+1} in (3.1). Because we prefer to work with the canonical bundle and not its conjugate, we start, as in the Proposition, by defining a vector field ζ and then let $Y = \overline{\zeta}$. Towards this end, let θ be a nowhere zero one-form annihilating $T^{1,0} \oplus \overline{T^{1,0}}$. Then

$$\theta \wedge df_1 \wedge \ldots \wedge df_n$$

is a nowhere zero section of the canonical bundle. This bundle is one dimensional, so

$$\omega = f\theta \wedge df_1 \wedge \ldots \wedge df_n.$$

Define ζ by

$$f\theta(\zeta) = 1, \quad df_j(\zeta) = 0, \quad d\overline{f}_j(\zeta) = 0$$

ζ can be thought of as a complex version of the Reeb vector field. In particular, it is transverse to $T^{1,0} \oplus T^{0,1}$.

We have

$$\begin{aligned} £_\zeta \omega &= d(i_\zeta \omega) + i_\zeta d\omega \\ &= d(f\theta(\zeta))df_1 \wedge \ldots \wedge df_n) + i_\zeta d\omega \\ &= 0. \end{aligned}$$

Lemma 3.1. *If $£_\zeta \omega = 0$ then $£_\zeta T^{0,1} = T^{0,1}$.*

Proof. We have for all vector fields ζ and v and all forms ω

$$£_\zeta i_v \omega = i_{£_\zeta v}\ \omega + i_v £_\zeta \omega.$$

So, if $v \in T^{0,1}$, hence $i_v \omega = 0$, and $£_\zeta \omega = 0$, then

$$i_{£_\zeta v}\ \omega = 0$$

and so $£_\zeta v$ is also in $T^{0,1}$. □

This Lemma has a partial converse: If $\pounds_\zeta T^{0,1} = T^{0,1}$ then $\pounds\omega = \alpha\omega$ for some function α.

Finally, we set $Y = \overline{\zeta}$. Thus, Y is transverse to $T^{1,0} \oplus T^{0,1}$ and

$$\pounds_Y T^{1,0} = \overline{\pounds_\zeta T^{0,1}} = T^{1,0}$$

and we are done.

References

1. T. Akahori, *A new approach to the local embedding theorem of CR-structures for $n \geq 4$ (the local solvability for the operator ∂ - b in the abstract sense)*, Mem. Amer. Math. Soc. 67 (1987), no. 366
2. H. Jacobowitz, "The canonical bundle and realizable CR hypersurfaces", Pacific J. of Math **127** (1987), 91-101.
3. H. Jacobowitz, *An Introduction to CR Structures*, Mathematical Surveys and Monographs **32** A.M.S., 1990.
4. H. Jacobowitz and F. Treves, "Nowhere solvable homogeneous partial differential equations", Bull. Amer. Math. Soc. (N.S.) **8** (1983), no. 3, 467-469.
5. M. Kuranishi, "Strongly pseudoconvex CR structures over small balls. III. An embedding theorem", Ann. of Math. (2) 116 (1982), no. 2, 249 - 330.
6. A. Newlander and L. Nirenberg, "Complex analytic coordinates in almost complex manifolds", Ann. of Math. (2) 65 (1957), 391 - 404.
7. L. Nirenberg, "On a question of Hans Lewy", Russian Math. Surveys **29** (1974), no. 2, 251-262.
8. I. Robinson and A. Trautman, "Conformal geometry of flows in N dimensions", J. Math. Phys. **24** (1983), 1425-1429.
9. I. Robinson and A. Trautman, "Cauchy-Riemann Structures in Optical Geometry", in *Proceedings of the Fourth Marcel Grossman Meeting on General Relativity*, Elsevier, 1986, 317-324.
10. A. Trautman, "On Complex Structures in Physics", in *On Einstein's path*, Springer, New York, 1999, 487 -501.
11. S. Webster, "On the proof of Kuranishi's embedding theorem", Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (1989), no. 3, 183 - 207.

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