THE TRAUTMAN CONJECTURE

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Conjecture (A. Trautman, 1998)

A three-dimensional CR manifold is locally realizable if and only if its canonical bundle admits a closed nowhere zero section.

CR structure : (M^3, H, J) with

- $H \subset TM$
- $J : H \to H$
- $J^2 = -I$

It is useful to extend J by complex linearity to a map

$$J : \mathbf{C} \otimes H \to \mathbf{C} \otimes H.$$

$$T^{1,0}(M) = \{v \in \mathbf{C} \otimes H : Jv = iv\}$$

Note

$$T^{1,0} \cap T^{0,1} = \{0\}$$

An equivalent definition of a CR structure on a three-dimensional manifold may be given in terms of a complex line bundle (M, B):

- $B \subset \mathbf{C} \otimes TM$
- $B \cap \bar{B} = \{0\}$

Then

$$B = T^{1,0}(M) \text{ and } Jv = iv \text{ for } v \in B$$

Example (Induced CR structure)

Let M^3 be a real hypersurface in $\mathbf{C}^2$ and let J denote the usual operator on $\mathbf{R}^4$ giving the complex structure.

$$\mathbf{R}^4 = (x_1, y_1, x_2, y_2)$$

$$J\partial_{x_k} = \partial_{y_k} \text{ and } J\partial_{y_k} = -\partial_{x_k}$$

Set
$H_p = T_pM \cap JT_pM$ for each $p \in M^3$.
J acts on H and (M, H, J) is a CR structure.

Or

Example (Induced CR structure, continued)

Let

$$\begin{aligned} B &= \{Z = \alpha\partial_{z_1} + \beta\partial_{z_2} \ : \ Z \in \mathbf{C} \otimes TM\} \\ &= T^{1,0}(\mathbf{C}^2) \cap \mathbf{C} \otimes TM \end{aligned}$$

The canonical bundle

:

$$\Omega = \{\omega \in \mathbf{C} \otimes \Lambda^2(TM) \ : \ i_b\omega = 0 \text{ for all } b \in \bar{B}\}.$$

The interior product $i_b\omega$ is given by $i_b\omega(X) = \omega(b \wedge X)$.

Example

For $M^3 \subset \mathbf{C}$

$$\Omega = \{\alpha dz_1 \wedge dz_2\}$$

CR functions and realizability

A function on the open set $U \subset M$ is a **CR function** if

$$\bar{Z} f = 0 \text{ for all } Z \in B = T^{1,0}(M).$$

(M,H,J) is **realizable** in a neighborhood of p if there exist CR functions f_1 and f_2 such that

$$F : M \to \mathbf{R}^4$$
$$x \to (\Re f_1, \Im f_1, \Re f_2, \Im f_2)$$

is an embedding.

- The induced CR structure is isomorphic to (m, H, J).
- It suffices to require $df_1 \wedge df_2 \neq 0$.

The general realizability problem is subtle. There are two easy results.

Proposition

A real analytic CR manifold M^{2n+1} is locally realizable in $\mathbf{C}^{n+1}$.

Proposition (The existence of a transverse action)

A CR manifold M^{2n+1} admitting a real vector field transverse to H and preserving the CR structure is locally realizable.

And one general result.

Proposition

Almost all three-dimensional CR manifolds are nowhere locally realizable.

Now for the physics:
A shear-free congruence of null geodesics on a four-dimensional Lorentz manifold M^4 induces a CR structure on the quotient manifold M^3. A congruence of null geodesics is just a foliation of M^4 by null geodesics.

- Use a congruence of null geodesics to construct a 2-dimensional complex bundle N on M^4 .
- Use shear-free to show that N projects to a CR structure on some M^3.

g is the Lorentz metric on M^4.
k the null vector field, $g(k,k) = 0$.
K be the real line bundle generated by k.

$$K_p^{\perp} = \{v \in T_pM : g(v,k) = 0\}.$$

Note that $K \subset K^{\perp}$ and that $K^{\perp}/K$ is an R^2 bundle on M^4.

Lemma

The metric g induces a well-defined positive definite inner product on $K^{\perp}/K$ and thus a J operator.

Proof
If $s \in K^{\perp}/K$ then $s = \tilde{\pi}_*(\sigma + \alpha k)$ for some $\sigma \in K^{\perp}$ and all $\alpha \in \mathbf{C}$.
Set $|s|^2 = g(\sigma + \alpha k, \sigma + \alpha k)$.

For $n \in \mathbf{C} \otimes K^{\perp}$, $[n]$ is the equivalency class of n in $\mathbf{C} \otimes K^{\perp} / \mathbf{C} \otimes K$.

$$N = \{n \in \mathbf{C} \otimes K^{\perp} : J[n] = -i[n]\}.$$

N is a two-dimensional complex vector bundle on M^4.
We have

$$N \subset \mathbf{C} \otimes K^{\perp} \subset \mathbf{C} \otimes TM^4$$

and

$$N \cap \bar{N} = \mathbf{C} \otimes K, \qquad N + \bar{N} = \mathbf{C} \otimes K^{\perp}.$$

Now consider the flow generated by the vector field k. (So far, we have only assumed the k is null). The orbit space is a three-dimensional manifold M^3.
$\pi \ : \ M^4 \to M^3$ and $\pi_* \mathbf{C} \otimes TM^4 \to \mathbf{C} \otimes TM^3$.
Let $p \in M^4$ and so $\pi_* N_p$ is a complex line in $\mathbf{C} \otimes TM^3$ at $\pi(p)$.
But

$$\pi(p) = \pi(q) \not\Rightarrow \pi_* N_p = \pi_* N_q.$$

Without additional assumptions on k the bundle N does not project to a well-defined subbundle of $\mathbf{C} \otimes TM^3$.

Definition

The null vector field k is said to be conformally geodesic if the associated flow preserves $K^{\perp}$.

The flow condition may be rewritten as

$$\mathcal{L}_k K^{\perp} \subset K^{\perp}.$$

Let $g(k)$ be the one-form defined by $g(k)v = g(k, v)$.
The flow condition may also be rewritten as

$$g(k) \wedge \mathcal{L}_k g(k) = 0$$

The Lorentz metric g induces a degenerate inner product on $K^{\perp}$ and therefore also a (degenerate) conformal structure.

Definition

A conformally geodesic vector field is shear-free if the associated flow preserves the conformal structure of $K^{\perp}$.

The physical hypothesis that k generates a shear-free congruence of null geodesics also can be formulated in terms of the Lie derivative.
A vector field k on a manifold M^4 with Lorentz metric g generates a shear-free congruence of null geodesics if and only if

$$\begin{aligned} g(k,k) &= 0 \\ g(k) \wedge \mathcal{L}_k g(k) &= 0 \\ \mathcal{L}_k g &= \lambda g + \phi \otimes g(k) \end{aligned}$$

where λ is a function, ϕ is a one-form, $g(k)$ is as defined above, and $\phi \otimes g(k)$ signifies the symmetric product constructed from the one-forms.

Recall the map of M^4 to the orbit space

$$\pi : M^4 \to M^3.$$

Lemma

For a shear-free congruence of null geodesics, $\pi_*(N)$ is a complex line bundle $\bar{B} \subset \mathbf{C} \otimes TM^3$ which satisfies $B \cap \bar{B} = \{0\}$.

So B defines a CR structure, $B = T^{1,0}(M^3)$.
That is, the physical assumptions lead to a CR structure on the orbit space.

The same conditions provide a two-form F associated to N which itself also passes down to M^3. The interest in such a two-form comes from considerations of Maxwell's equations. In classical physics, the components of the magnetic and electrical fields can be used to construct a real two-form F, called the Faraday tensor. Then, in the absence of charge, Maxwell's equations become $dF = 0$.

Define F by

$$F = \{\omega \in \mathbf{C} \otimes \Lambda^2 M^4 : i_v = 0 \text{ for all } v \in N\}$$

Note the similarity to the definition of the canonical bundle:

$$\Omega = \{\omega \in \mathbf{C} \otimes \Lambda^2(TM) \ : \ i_b\omega = 0 \text{ for all } b \in \bar{B} = T^(0,1)\}.$$

Further,

$$t \in N \Rightarrow \pi_* t \in \bar{B} = T^{0,1}(M^3)$$

F gives a well-defined two-form on M^3. Call this form F'. For $t' \in \bar{B} = T^{0,1}(M^3)$

$$t \in T^{0,1}(M^3) \Rightarrow i_t F' = 0.$$

Hence F' is section of the canonical bundle of M^3 and is nowhere zero.
In summary, the local quotient of a Lorentzian manifold under a shear-free congruence of null geodesics is a CR manifold which has a nowhere zero section of its canonical bundle.
This section being closed is related to Maxwell's equations and so is a reasonable hypothesis for physicists.
We repeat Trautman's conjecture.

Conjecture

If a CR manifold M^3 admits a nowhere zero closed section of its canonical bundle, then the CR structure is locally realizable.

A weak version of the conjecture is true and holds for all dimensions. Functions satisfying

$$df_1 \wedge \ldots \wedge df_k \neq 0$$

are called independent.
Functions satisfying

$$df_1 \wedge \ldots \wedge df_k \wedge d\bar{f}_1 \wedge \ldots \wedge d\bar{f}_k \neq 0$$

are called strongly independent.

Example

The hyperquadric $Q^3 \subset \mathbf{C}^2$ is defined by $\Im z_2 = |z_1|^2$. The bundle $T^{0,1}$ is generated by

$$L = \partial_{\bar{z}_1} - iz\partial_u$$

where $u = \Re z_2$. The CR function $f = z$ is strongly independent. The CR function $f = u + i|z|^2$ is independent, but not strongly independent (at the origin).

The following theorem preceded the formulation of Trautman's Conjecture and establishes a weak form.

Theorem (1987)

If the CR structure M^{2n+1} has n strongly independent CR functions near p and if the canonical bundle has a closed nowhere zero section then M^{2n+1} is realizable in a neighborhood of p.

The proof depends on the following complex version of the Transverse Action Proposition.

Proposition

M is realizable in a neighborhood of p if and only if there exists a complex vector field Y near p such that

- *Y is transverse to $T^{1,0} \oplus T^{0,1}$*
- *$\mathcal{L}_Y T^{1,0} = T^{1,0}$.*

Thus the existence of a real vector field such that $\mathcal{L}_v T^{1,0} = T^{1,0}$ is very special (since most realizable CR structures do not have such a vector field) but the existence of such a complex vector field characterizes realizability.

A possible counter-example

Find a one-form α on $\mathbf{R}^3$ such that

- if X is a real vector field and $i_X d\alpha = 0$ then $X = 0$ at the origin.
- if there exist a complex vector field Y and a function λ with

$$d(i_Y d\alpha) = \lambda d\alpha$$

then

$$(i_Y d\alpha) \wedge d\alpha = 0$$

at the origin.

Why might such an α exist?
Recall the famous result of Hans Lewy:
For most functions f, the equation

$$L = \partial_{\bar{z}} - iz\partial_u = f(z, \bar{z}, u)$$

does not have any solution in a neighborhood of any point.
This was modified by Nirenberg:
There exist functions f and g such that any solution to

$$\left(\partial_{\bar{z}} + f\partial_z - i(z+g)\partial_u\right)h = 0$$

has $dh(0) = 0$.
This proved there exist non-realizable CR structures.